Lecture Notes – Topics in General Relativity II

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Introduction

These lecture notes cover the content of the 8-hour short course Topics in General Relativity II held at the Gran Sasso Science Institute (GSSI) in spring 2026. The course was aimed at doctoral students in Mathematics.

The lecture notes focus on the basics of Lorentzian geometry and on the analysis of the wave equation on black hole spacetimes, motivated by the hyperbolic nature of the Einstein equations.

The notes are intended for a broad audience and provide an overview on some topics, rather than delving into technical details, for which references to the literature are suggested. Some exposure to the analysis of partial differential equations (as for instance the material covered in the first-semester courses at GSSI) is assumed.

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Syllabus

General Relativity, dating back to Einstein's 1915 proposal, is the most accurate and successful classical theory of gravitation in modern physics. Its core idea is that *gravity is the manifestation of the curvature of the "manifold" describing the space and time in which we live*.

$$\underset{\mathbb{R}}{\text{time}} + \underset{\mathbb{R}^3}{\text{space}} \xrightarrow[\text{in}]{\text{encoded}} \underset{(\mathcal{M}^{1+3}, g)}{\text{Spacetime}}$$

In these lecture notes we will discuss the following topics:

(a) What is a spacetime (aka Lorentzian manifold)?
(b) Can we make sense of a "time direction" in the spacetime?
(c) How to define the notion of curvature?
(d) Can we assign the notion of "trajectory" (or worldline) of observers?
(e) General Relativity is not only about Lorentzian geometry. Can we identify a class of physically relevant spacetimes?

1 Lorentzian Geometry and Causality

In the following, we take $n \in \mathbb{N}$, $n \geq 1$. Unless specified, we assume the main quantities to be smooth (i.e. C^∞).

Definition 1.1 (Lorentzian inner product) Let V be a $(1+n)$-dimensional $\mathbb{R}$-vector space. A *Lorentzian inner product* m is a map

$$m : V \times V \longrightarrow \mathbb{R}$$

such that

(a) m is symmetric, i.e. $m(v, w) = m(w, v)$ for every $v, w \in V$,
(b) m is bilinear, i.e. $m(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 m(v_1, w) + \lambda_2 m(v_2, w)$ for every $\lambda_1, \lambda_2 \in \mathbb{R}$, for every $v_1, v_2, w \in V$,
(c) m is non-degenerate, i.e. given $v \in V$, if $m(v, w) = 0$ for every $w \in V$, then $v = 0$,
(d) if $W \subseteq V$ is such that $m|_W$ is positive-definite (meaning that $m(w, w) > 0$ for every $0 \neq w \in W$), then $\dim W \leq n = \dim V - 1$ and equality is attained for some W.

Remark 1.2 (Euclidean inner product) Recall that an (Euclidean, real) inner product $\langle , \rangle : V \times V \to \mathbb{R}$ is symmetric, bilinear, it is such that $\langle v, v \rangle \geq 0$ for every $v \in V$ and moreover $\langle v, v \rangle = 0$ if and only if $v = 0$.

Proposition 1.3 (Classification of bilinear forms) *If m is a Lorentzian inner product on V, then there exists a basis $\{e_0, e_1, \dots, e_n\}$ in V such that:*

$$\begin{aligned} m(e_0, e_0) &= -1, \\ m(e_i, e_i) &= 1, && \forall\, i = 1, \dots, n, \\ m(e_\mu, e_\nu) &= 0, && \forall\, \mu, \nu = 0, \dots, n, \text{ with } \mu \neq \nu. \end{aligned}$$

Remark 1.4 (Bilinear forms and tensor products) Let V be a vector space. Given $\omega, \eta \in V^* = \{\varphi : V \to \mathbb{R}, \varphi \text{ linear}\}$, we define their tensor product as

$$\begin{aligned} \omega \otimes \eta : V \times V &\longrightarrow \mathbb{R} \\ (v_1, v_2) &\mapsto \omega(v_1)\eta(v_2). \end{aligned}$$

Since ω, η are linear, it follows that $\omega \otimes \eta$ is a bilinear form. Each tensor product is associated to a bilinear form.

Moreover, each bilinear form is associated to a tensor product as above. Indeed: let $\{e_0, \dots, e_n\}$ be a basis of V, let $\{e^0, \dots, e^n\}$ be the dual basis of V^*, i.e. $e^i(e_j) = \delta^i_j$, where δ is the Kronecker delta. Given a bilinear form m, we have a map

$$m \mapsto \sum_{i,j} m(e_i, e_j) e^i \otimes e^j.$$

Note that m is uniquely determined by its values on a basis of V.

In particular, we have a canonical isomorphism between 2-tensor products on V and bilinear forms on V.

Corollary 1.5 (Tensorial expression of the Lorentzian inner product) *Let $\{e_0, e_1, \dots, e_n\}$ be the basis for V from proposition 1.3 . By fixing an order on such a basis, we consider the associate coordinate functions $\{x^0, x^1, \dots, x^n\}$ on V. Let $\{dx^0, dx^1, \dots, dx^n\}$ be the respective dual basis on V^*. Then:*

$$\begin{aligned} m &= -dx^0 \otimes dx^0 + \sum_{i=1}^{n} dx^i \otimes dx^i \\ &= -\left(dx^0\right)^2 + \sum_{i=1}^{n} \left(dx^i\right)^2, \end{aligned}$$

where, in the last line, we introduced a common shorthand notation. This notation will be used over and over in these notes, also for general manifolds. It is an easy exercise to test this definition on the $\{e_0, \dots, e_n\}$ basis, so to verify that m coincides with the expression of proposition 1.3.

Definition 1.6 (Timelike, spacelike, null vectors) Let $v \in V$. Depending on the value of $m(v, v)$, we denote the vector as follows:

A classification of vectors stemming from a non-positive definite bilinear form

- if $m(v, v) < 0$, then v is *timelike*,

- if $v \neq 0$ and if $m(v,v) = 0$, then v is *null* (or lightlike),
- if $m(v,v) > 0$, then v is *spacelike*.

By convention, the zero vector is spacelike. We say that v is *causal* if it is either timelike or null.

Example 1.7 In 1 + 1 dimensions, take $V = \mathbb{R}^2$ with $\{x^0, x^1\}$ Cartesian coordinates. Let $\{e_0, e_1\}$ be the related orthonormal basis. Then, vectors are classified as in fig. 1.

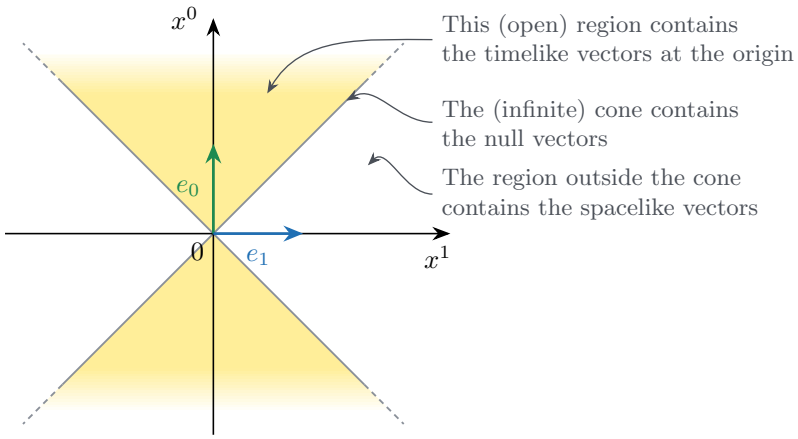


Figure 1: Causal and spacelike vectors in two dimensions.

See fig. 2 for the higher-dimensional generalization.

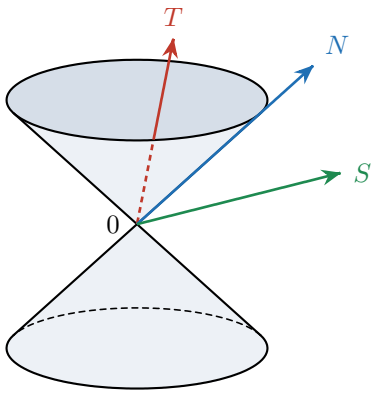


Figure 2: Causal and spacelike vectors in three dimensions.

Definition 1.8 (Timelike, spacelike, null subspaces) A subspace $W \subset V$ is said to be:

$$\begin{cases} \textit{spacelike} \text{ if } m|_W \text{ defines a Euclidean inner product,} \\ \textit{timelike} \text{ if } m|_W \text{ defines a Lorentzian inner product,} \\ \textit{null} \text{ if } m|_W \text{ defines a degenerate inner product.} \end{cases}$$

Remark 1.9 (Smooth manifolds and related notation) In the following, $\mathcal{M}$ will denote a smooth manifold, i.e. a topological space which is Hausdorff, second countable and admits an (equivalence class of) atlas(es) $\varphi_i : U_i \subset \mathcal{M} \longrightarrow V_i \subset \mathbb{R}^{1+n}$, where φ_i are homeomorphisms, U_i, V_i are open and the transition functions $\varphi_i \circ \varphi_j^{-1}$ are smooth $\forall\, i, j$.

Summary on smooth manifolds

In these notes we will mostly consider the case

$$\mathcal{M} \underset{\text{diffeomorphism}}{\cong} \mathbb{R}^{1+n} \quad \text{or} \quad \mathcal{M} \cong W,$$

where W open subset of $\mathbb{R}^{1+n}$. Given $p \in \mathcal{M}$,

$$T_p\mathcal{M} = \left\{\text{smooth curves } \gamma : (-\varepsilon, \varepsilon) \to \mathcal{M} \;\middle|\; \gamma(0) = p,\ \varepsilon > 0\right\} / \sim$$

is the tangent space of $\mathcal{M}$ at p, where two curves $\gamma_0, \gamma_1 : (-\varepsilon, \varepsilon) \longrightarrow \mathcal{M}$ with $\gamma_0(0) = \gamma_1(0) = p$ are equivalent ($\gamma_0 \sim \gamma_1$) if, for a chart (U, φ) around p:

$$\frac{d}{dt}\Big|_{t=0} (\varphi \circ \gamma_0)(t) = \frac{d}{dt}\Big|_{t=0} (\varphi \circ \gamma_1)(t).$$

There exist alternative (equivalent) definitions of tangent spaces, see e.g. [3]. The notation $\mathfrak{X}(\mathcal{M})$ will denote the set of vector fields on $\mathcal{M}$.

Exercise 1.10 Let $V = \mathbb{R}^{1+2}$ be endowed with the Lorentzian inner product m. Let $W \subset V$ be a null subspace. What is the causal character (i.e. timelike / spacelike / null) of vectors in W? Let $z \in V$ be orthogonal to W (where v_1, v_2 are orthogonal if $m(v_1, v_2) = 0$). Is z null?

Definition 1.11 (Lorentzian metric, Spacetime) Let $\mathcal{M}$ be a smooth manifold. A Lorentzian metric g on $\mathcal{M}$ is a smooth assignment

Spacetimes

$$p \mapsto g_p,$$

where, for every p, $g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ is a Lorentzian inner product, with $T_p\mathcal{M}$ tangent space of $\mathcal{M}$ at p. The pair $(\mathcal{M}, g)$ is said to be a Lorentzian manifold, or spacetime.

Remark 1.12 By *smooth assignment*, in definition 1.11, we mean that for every $p \in \mathcal{M}$ we can choose a chart

$$\begin{aligned} \phi : U \subset \mathcal{M} &\to V \subset \mathbb{R}^{1+n} \\ q &\mapsto \phi(q) = x \end{aligned}$$

near p so that

$$\begin{aligned} g_{\mu\nu} \circ \phi^{-1} \equiv \left(g_{\phi^{-1}(\cdot)}\right)_{\mu\nu} : V &\to \mathbb{R} \\ x &\mapsto \left(g_{\phi^{-1}(x)}\right)_{\mu\nu} \end{aligned}$$

is a smooth function for every μ, ν.

Tensors in General Relativity

Remark 1.13 If $\mathcal{M} \ni p \mapsto \mathcal{T}|_p$ is a smooth assignment and if

$$\mathcal{T}: \underbrace{T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}}_{k \text{ times}} \to \mathbb{R}$$

is multilinear, then $\mathcal{T}$ is a tensor of rank k (or, more precisely, a $(0,k)$-tensor). See [3, Chapter 12] for more about tensors on manifolds.

Remark 1.14 Every smooth manifold admits a Riemannian metric. However, not all smooth manifolds admit a Lorentzian metric.

Example 1.15 S^2 does not admit a Lorentzian metric.

Criterion to define a Lorentzian metric on a smooth manifold

Remark 1.16 Using some notions of differential geometry, one could prove that, given a smooth manifold $\mathcal{M}$, the following are equivalent (see e.g. [5]):

(a) $\mathcal{M}$ admits a Lorentzian metric,
(b) $\mathcal{M}$ admits a non-vanishing vector field,
(c) $\mathcal{M}$ is either non-compact or $\mathcal{M}$ is compact with vanishing Euler characteristic.

Example 1.17 (Minkowski spacetime) Take $\mathcal{M} = \mathbb{R}^{1+n}$ with coordinates $\{x^0, x^1, \ldots, x^n\}$. We can denote the coordinate basis[1] by

$$\left\{\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\} = \{\partial_0, \partial_1, \ldots, \partial_n\}.$$

We denote the related dual basis by $\{dt = dx^0, dx^1, \ldots, dx^n\}$. Then $(\mathbb{R}^{1+n}, m)$, with

$$m = -\left(dx^0\right)^2 + \left(dx^1\right)^2 + \ldots + \left(dx^n\right)^2$$

Minkowski spacetime

is a spacetime, called *Minkowski spacetime*. We will use the following notation:

$$m_{\mu\nu} = m\left(\partial_\mu, \partial_\nu\right), \quad \mu, \nu \in \{0, 1, \ldots, n\}.$$

For instance:

$$m_{00} = \left(-\left(dx^0\right)^2 + \left(dx^1\right)^2 + \ldots + \left(dx^n\right)^2\right)(\partial_0, \partial_0) = -1.$$

Definition 1.18 (Time orientation) A *time orientation* is a choice of a global smooth timelike vector field T on a spacetime $(\mathcal{M}, g)$.

Definition 1.19 (Future, past) A timelike vector X at $p \in \mathcal{M}$ is *future-pointing* if

$$g|_p(X, T) < 0.$$

It is said to be *past-pointing* if $g|_p(X, T) > 0$.

[1] Note that, given $p \in \mathbb{R}^{1+n}$, there is an isomorphism

$$\mathbb{R}^{1+n} \overset{\psi}{\leftrightarrow} \mathrm{Der}(\mathbb{R}^{1+n}) = \{v|_p : C^\infty(\mathbb{R}^{1+n}) \to \mathbb{R} \text{ satisfying the Leibniz rule}\}.$$

In particular, each basis vector e_i, $i = 0, \ldots, n$ of $\mathbb{R}^{1+n}$ can be associated to the directional derivative ∂_i [3, p.53].

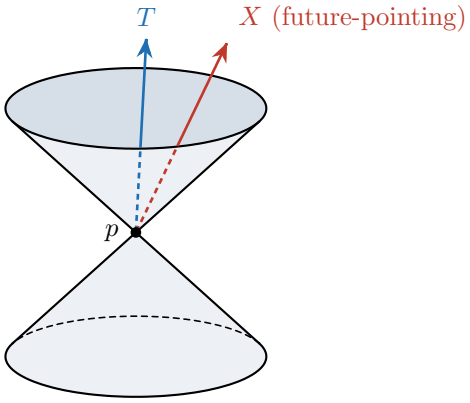


Figure 3: Future-pointing vectors with respect to a time orientation dictated by the timelike vector T.

A curve is said to be *timelike* and *future-directed* if its tangent vector field is timelike and future-pointing. Analogous definitions hold for the null and past-pointing case.

Future and past of spacetime events

The *chronological future* of $p \in \mathcal{M}$ is the set

$$I^+(p) = \{q \in \mathcal{M} : \exists \text{ a future-directed timelike curve from } p \text{ to } q\}.$$

The *causal future* of $p \in \mathcal{M}$ is the set

$$J^+(p) = \{p\} \cup \{q \in \mathcal{M} : \exists \text{ a future-directed causal curve from } p \text{ to } q\}.$$

Analogous definitions hold for the chronological and the causal past.

Remark 1.20 Analogous definitions hold for subsets of the spacetime. Given $\Sigma \subset \mathcal{M}$:

$$J^+(\Sigma) = \bigcup_{p\in\Sigma} J^+(p), \quad I^+(\Sigma) = \bigcup_{p\in\Sigma} I^+(p).$$

Example 1.21 Given $p_1, p_2 \in (\mathbb{R}^{1+1}, m)$ (2-dimensional Minkowski spacetime), $I^+(p_1)$ is open and $J^+(p_2)$ is closed (see fig. 4).

Exercise 1.22 Is the chronological future of a point always open? Is the causal future of a point always closed? (Hint: think about the case $(\mathcal{M} = \mathbb{R}^{1+1} \setminus \{0\}, m)$).

Remark 1.23 One can prove that a smooth manifold admits a Lorentzian metric if and only if it admits a time-orientable Lorentzian metric.

Example 1.24 (Time-orientable metrics vs time orientation) Consider the manifold $\mathcal{M} = S^1 \times I$, where I is an interval, having the causal structure of fig. 5 [5, p.150]. This Lorentzian manifold is not time-orientable.

However, it admits a time-orientable Lorentzian metric (see fig. 6).

Note that:

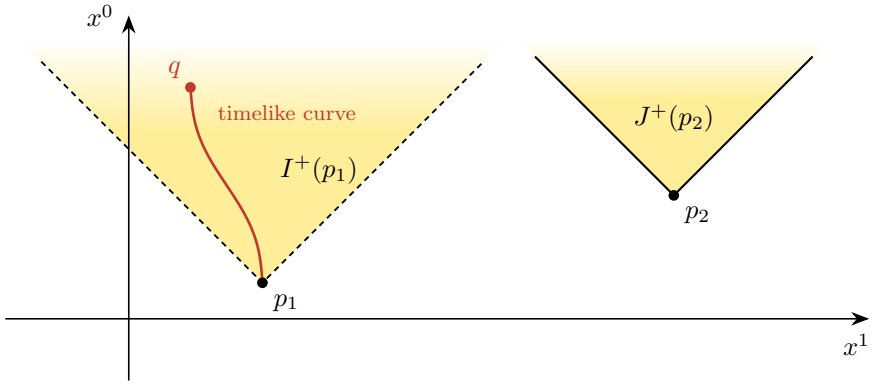


Figure 4: Chronological future and causal future.

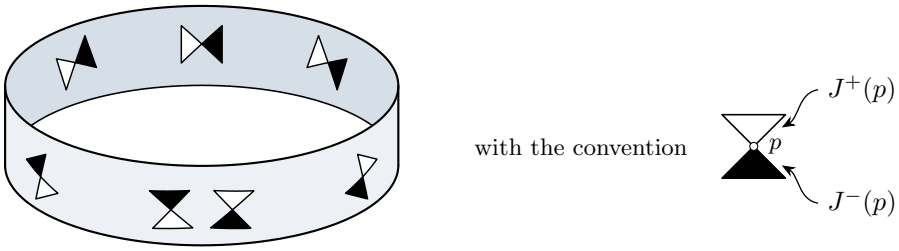


Figure 5: A Lorentzian manifold endowed with a metric that is not time-orientable. In particular, there exists no global smooth timelike vector field on it.

(a) The notions of orientability and time-orientability are unrelated: for instance one can define a time-orientable Lorentzian metric on the Möbius band,
(b) However, there does exist a similarity with the notion of orientability in differential geometry: one can always obtain a new time-orientable spacetime from a non-time-orientable one by exploiting the universal cover of the manifold.

Definition 1.25 (Inextendible curve) A future-directed causal curve $\gamma : (a, b) \to \mathcal{M}$ (where, possibly, $a = -\infty$ or $b = +\infty$) is said to be *future-inextendible* if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset (a, b)$, $\lim_{n\to+\infty} x_n = b$, such that $(\gamma(x_n))_{n\in\mathbb{N}}$ does not converge.

Remark 1.26 The above is a topological criterion for inextendibility [5, p. 154]. Alternatively [2], one can define $\gamma : (a, b) \to \mathcal{M}$, where (a, b) is a finite interval, to be inextendible if for all $\tilde{\gamma} : \tilde{I} \to \mathcal{M}$ (in the same class of regularity) with $I \subset \tilde{I}$ and $\tilde{\gamma}|_I = \gamma$, we have $\tilde{I} = I$.

Definition 1.27 (Domain of dependence) The *future domain of dependence* of $S \subset \mathcal{M}$ is

$$D^+(S) = \{q \in \mathcal{M} \mid \text{ any past-inextendible causal curve starting at } q \text{ intersects } S\}.$$

Example 1.28 See fig. 7 for a depiction of the future domain of dependence

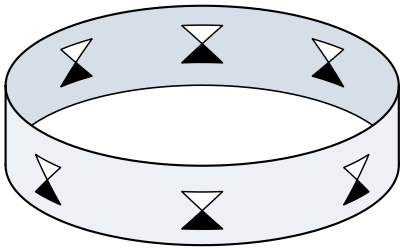


Figure 6: The same manifold of fig. 5, now endowed with a time-orientable Lorentzian metric.

of a subset of $(\mathbb{R}^{1+1}, m)$.

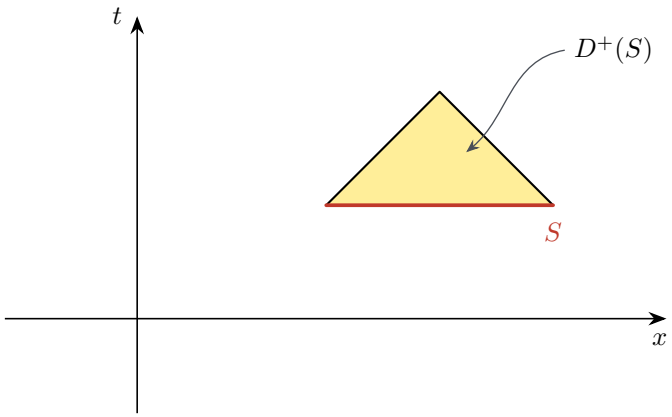


Figure 7: The domain of dependence of a compact subset S.

Definition 1.29 (Cauchy hypersurface) A *Cauchy hypersurface* is a spacelike hypersurface Σ such that all inextendible causal curves $\gamma : I \to \mathcal{M}$ intersect Σ once and only once.

Example 1.30 In $(\mathbb{R}^{1+1}, m)$, with $\{t, x\}$ coordinates, $\{t = 0\}$ is a Cauchy hypersurface. The hyperboloid $\{t^2 = x^2 + 1, t > 0\}$ is not a Cauchy hypersurface.

Definition 1.31 (Global hyperbolicity) A spacetime $(\mathcal{M}, g)$ is globally hyperbolic if it admits a Cauchy hypersurface.

Definition 1.32 (Lorentzian length) Let $\gamma : (s_1, s_2) \to \mathcal{M}$ be a causal curve, where $-\infty \leq s_1 < s_2 \leq +\infty$. Its Lorentzian length is defined as

$$L(\gamma) = \int_{s_1}^{s_2} \sqrt{-g(\dot{\gamma}(s), \dot{\gamma}(s))} ds.$$

Remark 1.33 We note the following:

- $L(\gamma) \geq 0$ for every causal curve γ. Null curves have zero length. For instance, take $\gamma : s \mapsto (s, s, \pi/2, 0)$ in (t, r, θ, φ) coordinates in $(\mathbb{R}^{1+3}, m = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2))$. Then, $L(\gamma) = 0$.

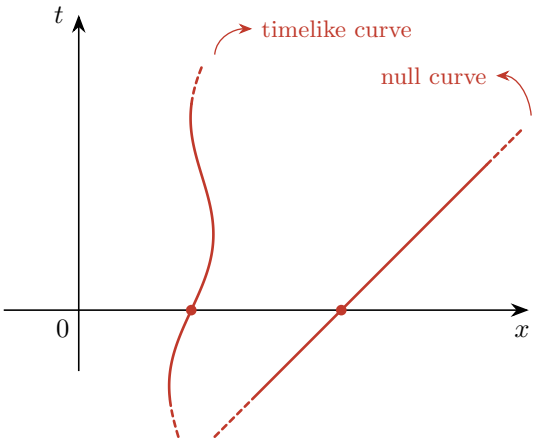


Figure 8: Every inextendible causal curve intersects $\{t = 0\}$ once and only once.

- When γ is a timelike curve, $L(\gamma)$ is interpreted as *proper time*, i.e. the time measured by the clock of an observer moving along the curve. Physically speaking, this is the quantity you can measure in experiments.
- Timelike geodesics locally maximize the proper time (*twin paradox*).

Differently from Riemannian geometry, here geodesics maximize (Lorentzian) lengths

Example 1.34 (Twin paradox) In $(\mathbb{R}^{1+1}, m)$, consider, in (t, x) coordinates, the curve

$$\gamma(s) = (s, 0), \qquad \text{for } s \in [0, 1],$$

and the curve given by the union of γ_1 and γ_2, informally denoted $\gamma_1 \cup \gamma_2$, where

$$\begin{cases} \gamma_1(s) = (s, s), & \text{for } s \in \left[0, \frac{1}{2}\right], \\ \gamma_2(s) = (s, 1 - s), & \text{for } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

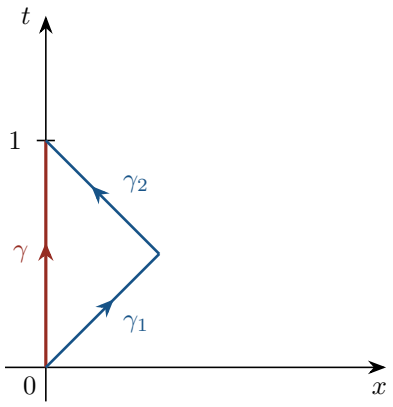


Figure 9: Two curves connecting the same endpoints.

Then:

$$L(\gamma) = \int_0^1 \sqrt{-m(\partial_t, \partial_t)}ds = 1,$$
$$L(\gamma_1 \cup \gamma_2) = \int_0^{\frac{1}{2}} \sqrt{-m(\partial_t + \partial_1, \partial_t + \partial_1)}ds +$$
$$+ \int_{\frac{1}{2}}^1 \sqrt{-m(\partial_t - \partial_1, \partial_t - \partial_1)}ds = 0 + 0 = 0.$$

This is related to the so-called *twin paradox* in Minkowski spacetime: timelike geodesics (such as γ) maximize the proper time. Any other causal curve from $(0,0)$ to $(0,1)$ that is not a geodesic (such as $\gamma_1 \cup \gamma_2$ or any small timelike smooth approximation of it) will have a smaller Lorentzian length.

In particular, we could modify $\gamma_1 \cup \gamma_2$ so that it becomes a smooth timelike curve having length $\epsilon > 0$ (see fig. 10).

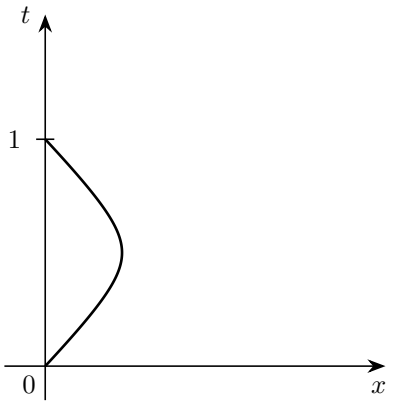


Figure 10: The curve $\gamma_1 \cup \gamma_2$ can be suitably approximated by a timelike curve having proper time $\epsilon > 0$, with ϵ small.

Remark 1.35 In a general spacetime, timelike geodesics maximize the proper time only locally. For instance think of $\mathcal{M}$ given by a cylinder, i.e. a subset of $(\mathbb{R}^{1+3}, m)$ with the induced metric (see fig. 11). In this case, both γ_1 and γ_2 are timelike geodesics connecting the same points, however they have different proper time.

2 Connections, Geodesics and Wave Equations on Lorentzian Manifolds

The Einstein equations can be cast, in a suitable coordinate system, as a system of non-linear wave equations. Wave equations on fixed backgrounds are therefore the simplest model to gain insights on non-linear problems in

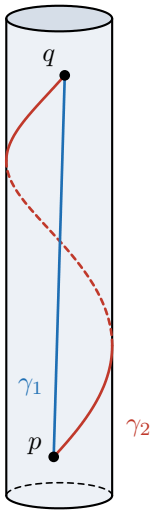


Figure 11: In a general spacetime, timelike geodesics maximize proper time only locally.

relativity. In the following, we introduce the main technical tools to study such partial differential equations on curved backgrounds. For the next sections, see also [1].

2.1 Affine Connections

How to compare vectors belonging to different tangent spaces

Definition 2.1 (Affine connection) An *affine connection* is a map

$$\begin{aligned} \nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) &\longrightarrow \mathfrak{X}(\mathcal{M}) \\ (X, Y) &\longmapsto \nabla_X Y \end{aligned}$$

such that:

(a) $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$, for every $X_1, X_2, Y \in \mathfrak{X}(\mathcal{M})$,
(b) $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$, for every $X, Y_1, Y_2 \in \mathfrak{X}(\mathcal{M})$,
(c) $\nabla_{fX} Y = f \nabla_X Y$, for every $f \in C^\infty(\mathcal{M})$, $X, Y \in \mathfrak{X}(\mathcal{M})$,
(d) $\nabla_X (fY) = X(f) Y + f \nabla_X Y$, for every $f \in C^\infty(\mathcal{M})$, $X, Y \in \mathfrak{X}(\mathcal{M})$.

We also say that $\nabla_X Y$ is the *covariant derivative* of Y with respect to X.

Remark 2.2 A similar definition holds for $\nabla_X T$, where T is a tensor, see remark 1.13. Note that a vector field is a rank-1 tensor.

Einstein, when asked what he liked best of Italy, supposedly used to say "Spaghetti and Levi-Civita!"

Remark 2.3 $(\mathcal{M}, g)$ always admits a *Levi-Civita connection*, i.e. the unique ∇ such that:

(a) ∇ is compatible with the metric:

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \qquad \forall X, Y, Z \in \mathfrak{X}(\mathcal{M})$$

(b) ∇ is symmetric:

$$\nabla_X Y - \nabla_Y X = [X, Y] = XY - YX.$$

Remark 2.4 (Interpretation of the affine connection) In $(\mathbb{R}^{1+n}, m)$, if ∇ is the Levi-Civita connection, $\nabla_X Y$ in Cartesian coordinates is just a directional derivative. For general manifolds, we interpret it as a map that allows to compare vectors belonging to different tangent spaces.

Remark 2.5 We summarize and provide complementary facts on the main geometric notions discussed so far.

(i) Given $p \in \mathcal{M}$, $T_p\mathcal{M}$ is a vector space having the same dimension of $\mathcal{M}$.

(ii) Given a vector $X|_p \in T_p\mathcal{M}$, one can choose a (local) coordinate basis

$$\{\partial_0, \partial_1, \dots, \partial_n\} \quad \text{in } T_p\mathcal{M}$$

and express X in this coordinate basis:

$$\begin{aligned} X|_p &= (X^0, X^1, \dots, X^n) \\ &= X^0\partial_0 + X^1\partial_1 + \cdots + X^n\partial_n. \end{aligned}$$

(iii) A vector field is a map

$$\begin{aligned} X : \mathcal{M} &\longrightarrow T\mathcal{M} = \bigsqcup_{p\in\mathcal{M}} T_p\mathcal{M} \\ p &\longmapsto (p, X|_p). \end{aligned}$$

We also write:

$$X \in \mathfrak{X}(\mathcal{M}).$$

(iv) We exploit the Einstein summation convention:

$$X = \sum_{\mu=0}^{n} X^\mu \partial_\mu = X^\mu \partial_\mu.$$

(v) Given a vectorfield $X : \mathcal{M} \to T\mathcal{M}$, there is an associated *covector* field

About raising and lowering indices

$$X^\flat : \mathcal{M} \longrightarrow T^*\mathcal{M},$$

where $X^\flat = X_\nu\, dx^\nu$ and $X_\nu = g_{\mu\nu}X^\mu$.

(vi) $\{dx^\nu\}_{\nu=0}^n$ is the dual basis to $\{\partial_\mu\}_{\mu=0}^n$ on $T^*\mathcal{M}$.

(vii) The metric g provides an isomorphism

$$\begin{aligned} \flat : T\mathcal{M} &\longrightarrow (T\mathcal{M})^* = T^*\mathcal{M} \\ X &\longmapsto \big(Y \mapsto g(X, Y)\big). \end{aligned}$$

The inverse map is $\sharp = \flat^{-1}$. This is called musical isomorphism.

(viii) Take $Y = \partial_\nu$ for some fixed $\nu \in \{0, \dots, n\}$. Let $X \in \mathfrak{X}(\mathcal{M})$. Then:

$$\begin{aligned} \nabla_Y X &= (\nabla_Y X)^\mu \partial_\mu = (Y^\alpha \nabla_\alpha X)^\mu \partial_\mu \\ &= (\delta^\alpha_\nu \nabla_\alpha X^\mu)\, \partial_\mu. \end{aligned}$$

Formally:

$$\nabla_\nu X = (\nabla_\nu X^\mu)\,\partial_\mu, \qquad \text{for some } \nu \in \{0, 1, \dots, n\}$$

Covariant derivatives in coordinates

This expression can be written in terms of the basis $\{\partial_\alpha\}_{\alpha=0}^n$:

$$\nabla_\nu X^\mu = \partial_\nu X^\mu + \Gamma^\mu_{\delta\nu} X^\delta \qquad \text{(when } X \text{ is a vector)}$$

where $\Gamma^\mu_{\delta\nu}$ are the Christoffel symbols

$$\Gamma^\mu_{\delta\nu} = \frac{1}{2} g^{\mu\alpha}\left(\partial_\delta g_{\alpha\nu} + \partial_\nu g_{\alpha\delta} - \partial_\alpha g_{\delta\nu}\right)$$

for every μ, δ, ν.

(ix) For a $(k, 0)$ tensor:

$$\nabla_\mu T^{\alpha_1\cdots\alpha_k} = \partial_\mu T^{\alpha_1\cdots\alpha_k} + \Gamma^{\alpha_1}_{\delta\mu} T^{\delta\alpha_2\cdots\alpha_k} + \Gamma^{\alpha_2}_{\delta\mu} T^{\alpha_1\delta\alpha_3\cdots\alpha_k} + \cdots + \Gamma^{\alpha_k}_{\delta\mu} T^{\alpha_1\cdots\alpha_{k-1}\delta}$$

(x) Given $\phi \in C^\infty(\mathcal{M})$, $X \in \mathfrak{X}(\mathcal{M})$, we write

$$\nabla_X \phi = \sum_{\mu=0}^n X^\mu \partial_\mu \phi = X^\mu \partial_\mu \phi = X \cdot \phi.$$

Note that ϕ is a rank-0 tensor field.

Definition 2.6 (Curvature) The curvature, or Riemann tensor, of $\mathcal{M}$ is the map

$$\begin{aligned} R : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) &\longrightarrow \mathfrak{X}(\mathcal{M}) \\ (X, Y, Z) &\longmapsto R(X, Y)Z \end{aligned}$$

where $R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$.

Remark 2.7 The above is denoted "tensor" because it can be seen as a smooth assignment of multilinear maps:

$$p \longmapsto R|_p$$

where (recall: $\mathfrak{X}(\mathcal{M}) = \{v : \mathcal{M} \to T\mathcal{M}\}$)

$$R|_p : T_p\mathcal{M} \times T_p\mathcal{M} \times T_p\mathcal{M} \times T_p^*\mathcal{M} \longrightarrow \mathbb{R}.$$

The spacetime of Special Relativity is flat

Example 2.8 In Minkowski spacetime $(\mathbb{R}^{1+3}, m)$, the Riemann tensor is identically zero (intuitively: no curvature entails no gravity). This can be seen in different ways.[2] For instance, take three general vector fields

$$X = \sum_\mu X^\mu \partial_\mu, \qquad Y = \sum_\nu Y^\nu \partial_\nu, \qquad Z = \sum_\alpha Z^\alpha \partial_\alpha,$$

with respect to Cartesian coordinates. Then, using the expressions in remark 2.5 and the fact that $\nabla_\nu = \partial_\nu$ in Cartesian coordinates:

$$\nabla_X(\nabla_Y Z)^\alpha = X^\mu \nabla_\mu (Y^\nu \nabla_\nu Z^\alpha) = X^\mu \partial_\mu (Y^\nu \partial_\nu Z^\alpha)$$

[2] Exercise: prove it in one line.

$$= X^\mu(\partial_\mu Y^\nu)\partial_\nu Z^\alpha + X^\mu Y^\nu \partial_\mu \partial_\nu Z^\alpha.$$

Similarly:

$$\nabla_Y(\nabla_X Z)^\alpha = \underbrace{Y^\nu(\partial_\nu X^\mu)}_{=\nabla_Y X^\mu}\partial_\mu Z^\alpha + X^\mu Y^\nu \partial_\mu \partial_\nu Z^\alpha.$$

So, using the symmetry of the Levi-Civita connection:

$$\nabla_X(\nabla_Y Z)^\alpha - \nabla_Y(\nabla_X Z)^\alpha = \left[(\nabla_X Y)^\nu - (\nabla_Y X)^\nu\right]\partial_\nu Z^\alpha = [X,Y]^\nu \partial_\nu Z^\alpha = \nabla_{[X,Y]} Z^\alpha,$$

yielding

$$R(X,Y)Z = 0, \qquad \forall X, Y, Z \in \mathfrak{X}(\mathcal{M}).$$

The following definition is analogous to that in the case of Riemannian geometry.

Definition 2.9 (Geodesic) A geodesic is a curve $s \mapsto \gamma(s)$ for which the vector field $\dot\gamma(s)$ is parallel-transported along γ, i.e. $\nabla_{\dot\gamma(s)}\dot\gamma(s) = 0$. We say that s is the affine parameter of the geodesic.

Example 2.10 In $(\mathbb{R}^{1+1}, m)$:

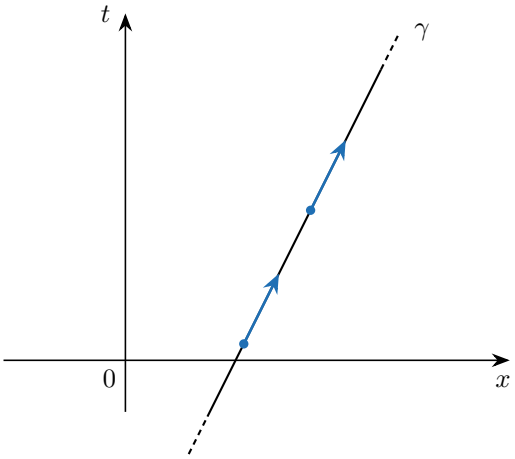


Figure 12: In Minkowski spacetime, straight lines are geodesic.

Note that, contrary to the Riemannian setting in $\mathbb{R}^2$, here timelike geodesics maximize the length, i.e. the proper time.

2.2 Postulates of General Relativity

The concept of geodesic is related to a crucial postulate in the theory of General Relativity.

Postulate 1 (Geodesic hypothesis)

> Free-falling *massive* test particles move along *timelike* geodesics.
> Free-falling *massless* particles move along *null* geodesics.

By "free-falling" we mean a body that is subject to gravity and no other forces are exerted on it.

Remark 2.11 The intuition behind the geodesic hypothesis is that the geodesic equation of General Relativity should be thought of as a generalization of $F = ma$. Given a particle having mass m and subject only to gravity via the gravitational potential ϕ, we can express its motion in terms of the second law of Newton. Analogously, given a free-falling particle and the Levi-Civita connection ∇ determined by a Lorentzian metric g, we can express the particle motion via the geodesic equation.

The geodesic equation generalizes $F = ma$

Newtonian gravity (Newton's second law)	$\longrightarrow$	General Relativity (Geodesic equation)
$m\frac{d^2x^i}{dt^2} = -m\partial_i\phi$		$\nabla_{\dot\gamma}\dot\gamma = 0$

So far, we considered general Lorentzian metrics on smooth manifolds. In practice, we are interested in physically relevant spacetimes. In other words, we are interested in solutions to the Einstein equations.

Postulate 2 (Einstein equations)

A Lorentzian metric g is physically relevant if it satisfies the Einstein equations:

$$\mathrm{Ric}(g) - \frac{1}{2}\mathrm{R}(g)g + \Lambda g = 8\pi T,$$

where $\mathrm{Ric}(g)$ is the Ricci tensor, R is its trace, i.e. the Ricci scalar. The constant $\Lambda \in \mathbb{R}$ is called the cosmological constant and T is the energy-momentum tensor, encoding the matter content of spacetime.

Remark 2.12 One could think of the Einstein equations as a generalization of the Poisson equation describing gravitational motion. In particular, given the gravitational potential ϕ and a matter mass density ρ in Newtonian gravity:

The Einstein equations generalize the Newtonian Poisson equation

Newtonian gravity (Poisson equation)	$\longrightarrow$	General Relativity (Einstein equations)
$\Delta\phi = 4\pi\rho$		$\mathrm{Ric}(g) - \frac{1}{2}\mathrm{R}(g)g + \Lambda g = 8\pi T.$

Note that the Poisson equation is an elliptic equation, whereas the Einstein equations are, after a choice of coordinates, a system of quasilinear wave equations.

2.3 The Wave Equation in General Relativity

We now introduce the concept of Laplace-Beltrami operator. In the following, $(\mathcal{M}, g)$ is a globally hyperbolic spacetime.

Definition 2.13 (Divergence, Laplace–Beltrami operator) Let $X \in \mathfrak{X}(\mathcal{M})$. We define its divergence as:

$$\mathrm{div}\, X := \nabla_\mu X^\mu.$$

Let $\phi \in C^\infty(\mathcal{M})$. We define the Laplace–Beltrami operator (or covariant wave operator) as:

$$\Box_g \phi := \operatorname{div}((d\phi)^\sharp) = \nabla_\mu \nabla^\mu \phi.$$

The equation $\Box_g \phi = 0$ is the covariant wave equation on $(\mathcal{M}, g)$.

Example 2.14 In $(\mathbb{R}^{1+3}, m)$, we have $\nabla_\mu = \partial_\mu$ for every μ, so:

$$\Box_m \phi = \nabla_\mu \nabla^\mu \phi = \partial_\mu \partial^\mu \phi = m^{\mu\nu} \partial_\mu \partial_\nu \phi = -\partial_t^2 \phi + \delta^{ij} \partial_i \partial_j \phi = -\partial_t^2 \phi + \Delta \phi.$$

The Laplace-Beltrami operator in Minkowski spacetime is just the d'Alembertian

The following definitions are inspired by Lagrangian field theories (see, e.g., [4, section 6.2]).

Definition 2.15 (Energy momentum tensor, energy current) The canonical energy-momentum tensor associated to a massless scalar field ϕ is:

$$T[\phi] := d\phi \otimes d\phi - \frac{1}{2} g^{-1}(d\phi, d\phi)\, g,$$

or, in coordinates:

$$T_{\mu\nu}[\phi] := \partial_\mu \phi\, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi\, \partial_\beta \phi.$$

Given $X \in \mathfrak{X}(\mathcal{M})$, we define the X-current of ϕ as

$$J^X_\mu[\phi] := T_{\mu\nu}[\phi] X^\nu.$$

Proposition 2.16 (divergence of T, divergence of J^X) *We have:*

$$\operatorname{div} T[\phi] = (\Box_g \phi)\, d\phi,$$

i.e.

$$\nabla^\mu T_{\mu\nu}[\phi] = (\Box_g \phi)\, \partial_\nu \phi.$$

The energy-momentum tensor of ϕ is divergenceless if ϕ solves the wave equation

In particular, if $\Box_g \phi = 0$, then T is divergenceless. Suppose that $\Box_g \phi = 0$, then, given $X \in \mathfrak{X}(\mathcal{M})$:

$$\nabla^\mu J^X_\mu[\phi] = K^X[\phi], \quad \textit{where}$$
$$K^X[\phi] := T^{\mu\nu}[\phi]\, \nabla_\mu X_\nu.$$

Remark 2.17 (Divergence theorem) Recall the divergence theorem in differential geometry: if $\mathcal{M}$ is a compact orientable manifold with boundary $\partial\mathcal{M}$, and if ε is a volume form on $\mathcal{M}$, then:

$$\int_{\mathcal{M}} (\operatorname{div} X)\, \varepsilon = \int_{\partial\mathcal{M}} g(X, N) \sigma, \qquad \text{for every } X \in \mathfrak{X}(\mathcal{M})$$

where N is the unit normal to $\partial\mathcal{M}$ and σ is the induced volume form on $\partial\mathcal{M}$. We recall that ε is a volume form if it is a differential form of degree n (where $n = \dim \mathcal{M}$) such that $\varepsilon(E_1, \ldots, E_n) = 1$, for every orthonormal frame $\{E_1, \ldots, E_n\}$ on $\mathcal{M}$.

Here, we say that $X, Y \in \mathfrak{X}(\mathcal{M})$ are orthogonal if $g(X,Y) = 0$. In our Lorentzian setting, it follows from our definitions that a null vector is orthogonal to itself. In particular, the divergence theorem above applies to our case, provided that we choose the unit normal vectors to the manifold depending on their causal nature. That is, we choose the convention for the normals to $\partial\mathcal{M}$ displayed in figure 13 (see also the discussion in [4, Chapter 5]).

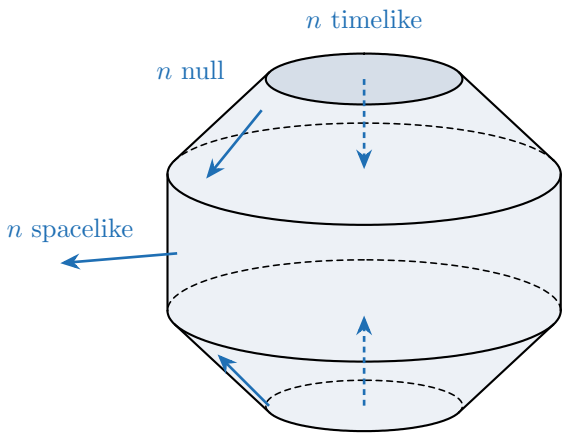


Figure 13: Convention for unit normals to a surface in the Lorentzian setting.

Remark 2.18 In the following, we will omit the volume form from the integrals of the main expressions. For instance, we will write

$$\int_{\mathcal{M}} \operatorname{div} X \qquad \text{in place of} \qquad \int_{\mathcal{M}} (\operatorname{div} X)\, \varepsilon.$$

Definition 2.19 (H^k norm) Given a C^∞ function $\phi\colon \mathbb{R}^{1+3} \to \mathbb{R}$, where Σ is a hypersurface, we define its H^1 norm as

$$\|\phi\|^2_{H^1(\Sigma)} := \int_\Sigma \left(\phi^2 + \sum_{\alpha=0}^{3} (\partial_\alpha \phi)^2 \right).$$

An analogous definition holds for the H^k norm, $k \in \mathbb{N}$, where derivatives up to the k-th order are allowed.

Theorem 2.20 (Energy boundedness) *Let Σ_0, Σ_τ be homologous (i.e. having same homology class) spacelike hypersurfaces with common boundary and let $X \in \mathfrak{X}(\mathcal{M})$ be timelike and future-directed. Let $\mathcal{R}_\tau$ be the spacetime region enclosed between Σ_0 and Σ_τ and assume that it can be foliated by analogous hypersurfaces. Assume that $\Box_g \phi = 0$.*

Then:

$$\int_{\Sigma_\tau} J^X_\mu[\phi]\, n^\mu = \int_{\Sigma_0} J^X_\mu[\phi]\, n^\mu - \int_{\mathcal{R}_\tau} K^X[\phi]. \tag{2.1}$$

Moreover, there exists $C > 0$ depending on the region $\mathcal{R}_\tau$ such that, for

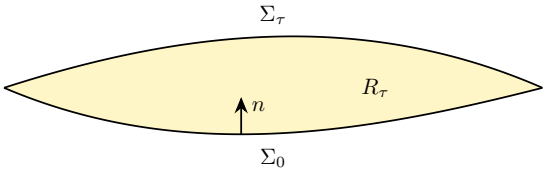


Figure 14: The setting of theorem 2.20.

$0 < \tau \leq 1$:

$$\|\phi\|_{H^1(\Sigma_\tau)} \leq C \, \|\phi\|_{H^1(\Sigma_0)}. \tag{2.2}$$

Proof. The divergence theorem applied to $\mathcal{R}_\tau$ gives

$$\int_{\mathcal{R}_\tau} \operatorname{div} J^X = \int_{\partial\mathcal{R}_\tau} g(J, n).$$

By prop. 2.16, the LHS above is $\displaystyle\int_{\mathcal{R}_\tau} K^X[\phi]$.

By construction, the RHS above is $\displaystyle\int_{\Sigma_0} J^X_\mu[\phi]\, n^\mu - \int_{\Sigma_\tau} J^X_\mu[\phi]\, n^\mu$. Hence, we obtained relation (2.1). Now, notice that

$$\begin{aligned} |K^X[\phi]| &= \left|T_{\mu\nu}[\phi]\nabla^\mu X^\nu\right| = \\ &= \left|\left(\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi\right)\nabla^\mu X^\nu\right| \\ &\leq c\sum_\alpha(\partial_\alpha\phi)^2 \;\leq\; \tilde{c}\, T_{\mu\nu}X^\mu n^\nu \;=\; \tilde{c}\, J^X_\mu n^\mu, \end{aligned}$$

where c above is independent of ϕ and τ and where the last inequality follows from [2, lemma 4.6]. We note that J^X is past-pointing. Indeed, recall that

$$J^X = (X\cdot\phi)d\phi - \frac{1}{2}g(d\phi, d\phi)X$$

Assume that X is timelike and $X = e^0$ (the case in which X is null follows similarly). In the orthonormal basis $\{X, e_1, \ldots, e_n\}$, we have:

$$d\phi = c^0 X + c^1 e_1 + \cdots + c^n e_n,$$

for some $c^i \in \mathbb{R}$, $i = 1, \ldots, n$. Moreover, we have $X\cdot\phi = g(X, d\phi)$. Then, J is causal since

$$\begin{aligned} g(J^X, J^X) &= \cancel{(X\cdot\phi)^2 g(d\phi, d\phi)} - \cancel{(X\cdot\phi)g(d\phi, d\phi)g(d\phi, X)} + \frac{1}{4}g(d\phi, d\phi)^2 g(X, X) \\ &= -\frac{1}{4}g(d\phi, d\phi)^2 \leq 0. \end{aligned}$$

This gives that J^X is past-pointing since

$$g(J^X, X) = g(X, d\phi)^2 - \frac{1}{2}g(d\phi, d\phi)g(X, X)$$

$$= \frac{(c^0)^2}{2} g(X,X)^2 - \frac{1}{2}\big((c^1)^2 g(e_1,e_1) + \cdots + (c^n)^2 g(e_n,e_n)\big) g(X,X)$$
$$= \frac{(c^0)^2}{2} + \frac{1}{2}\big((c^1)^2 + \cdots + (c^n)^2\big) \geq 0.$$

So, relation (2.1) and the bound on K^X above give

$$\int_{\Sigma_\tau} J^X_\mu[\phi]\, n^\mu = \int_{\Sigma_0} J^X_\mu[\phi]\, n^\mu - \int_{\mathcal{R}_\tau} K^X[\phi]$$
$$\leq \int_{\Sigma_0} J^X_\mu[\phi]\, n^\mu + C\int_0^\tau \Big(\int_{\Sigma_{\tau^*}} J^X_\mu[\phi]\, n^\mu\Big) d\tau^*,$$

for some $C > 0$ independent of τ, where the inequality above follows from the co-area formula, and from the fact that $J^X_\mu n^\mu = g(J^X, n) \geq 0$, i.e. J^X is past-pointing.

Now, define:

$$f(\tau) := \int_{\Sigma_\tau} J^X_\mu[\phi]\, n^\mu$$

Then, we found:

$$f(\tau) \leq f(0) + c\int_0^\tau f(\tau')\, d\tau'. \tag{2.3}$$

To bound the L^2 norm of ϕ, we first note that:

$$\int_{\mathcal{R}} \nabla_\alpha(\phi^2 X^\alpha) = \int_{\mathcal{R}} \big(2\phi X\cdot\phi + \phi^2 \nabla_\alpha X^\alpha\big)$$

Using the divergence theorem, the fact that $X^\alpha n_\alpha \sim -c$ for $c > 0$ (since n is future-pointing) and that $|\operatorname{div} X| \leq C$:

$$\int_{\Sigma_\tau} \phi^2 \leq \int_{\Sigma_0} \phi^2 + c\int_0^\tau \Big(\int_{\Sigma_{\tau'}} \phi^2\Big) d\tau' + c\int_0^\tau f(\tau')\, d\tau'.$$

If we add the above to (2.3), we find:

$$\Big[f(\tau) + \int_{\Sigma_\tau} \phi^2\Big] \leq \Big[f(0) + \int_{\Sigma_0} \phi^2\Big] + c\int_0^\tau \Big[f(\tau') + \int_{\Sigma_{\tau'}} \phi^2\Big] d\tau'.$$

Grönwall's inequality and the fact that[3]:

$$f(\tau) \sim \int_{\Sigma_\tau} \sum_{\alpha=0}^n (\partial_\alpha \phi)^2$$

yield:

$$\|\phi\|_{H^1(\Sigma_\tau)} \leq Ce^{C\tau} \|\phi\|_{H^1(\Sigma_0)}, \qquad \forall\, 0 < \tau \leq 1,$$

for some $C > 0$. □

Remark 2.21 The above result holds for a general timelike vector field X. Refinements are possible under additional assumptions on X.

[3]Here $A \sim B$ if both $A \leq c_1 B$ and $A \geq c_2 B$ hold. The lower bound for f was discussed above, whereas the upper bound follows by definition.

Remark 2.22 We proved a priori estimates for the solution ϕ. Analogous arguments allow to prove well-posedness for ϕ when the initial data are prescribed on a Cauchy hypersurface. See [2, Theorem 4.18].

Let us now focus on the case of the (1+3)-dimensional Minkowski space-time

$$\left(\mathbb{R}^{1+3},\ m = -dt^2 + (dx^1)^2 + \cdots + (dx^3)^2\right).$$

Let $X = \partial_t$. Consider the Cauchy hypersurface $S_0 = \{t = t_0\}$ and let B_0 be the ball of radius R on it:

$$B_0 = \left\{t = t_0,\ (x^1)^2 + (x^2)^2 + (x^3)^2 \leq R^2\right\}.$$

Let $S_1 = \{t = t_1\}$ and $B_1 = D^+(B_0) \cap S_1$, with $t_1 > t_0$.

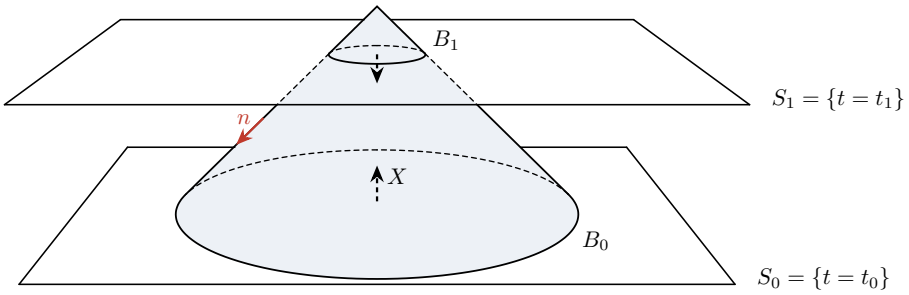


Figure 15: The Minkowski setting for the energy of linear waves.

We define the energy of ϕ as:

$$E[\phi](t) := \int_{\{t\}\times\mathbb{R}^3} g\,(J, \partial_t) = \int_{\{t\}\times\mathbb{R}^3} T_{00}[\phi] = \int_{\{t\}\times\mathbb{R}^3} \frac{1}{2}\Big[(\partial_t\phi)^2 + \delta^{ij}\partial_i\phi\,\partial_j\phi\Big].$$

Theorem 2.23 (Energy identity) *Let ϕ solve*

$$\begin{cases} \Box_m\phi(t,x) = 0, & \forall\,(t,x) \in \mathbb{R}^{1+3}, \\ \phi|_{S_0}(x) = \phi_0(x), & \forall\, x \in \mathbb{R}^3, \\ (X\cdot\phi)|_{S_0}(x) = \phi_1(x), & \forall\, x \in \mathbb{R}^3, \end{cases}$$

for some $\phi_0, \phi_1 \in H^k(\mathbb{R}^3)$ for some large $k \in \mathbb{N}$. Then:

$$E[\phi](t_1) = E[\phi](t_0) \qquad \forall\, t_1 > t_0.$$

Energy conservation in the Minkowski case, using a geometric approach

Proof. One could prove it by multiplying both sides of $\Box_m\phi(t,x) = 0$ by $\partial_t\phi$ and integrating by parts. However, we will follow a more geometric approach. Consider the region $\mathcal{R}$ displayed in figure 16.

Let us apply the divergence theorem on $\mathcal{R}$ for $\operatorname{div} J^X_\mu$, where $X = \partial_t$:

$$\int_{B_1} g\,(J, \partial_t) = \int_{B_0} g\,(J, \partial_t) + \int_C g(J, n),$$

where C denotes the null boundary of the truncated cone. Without loss of generality (we can ultimately apply a density argument) we can assume

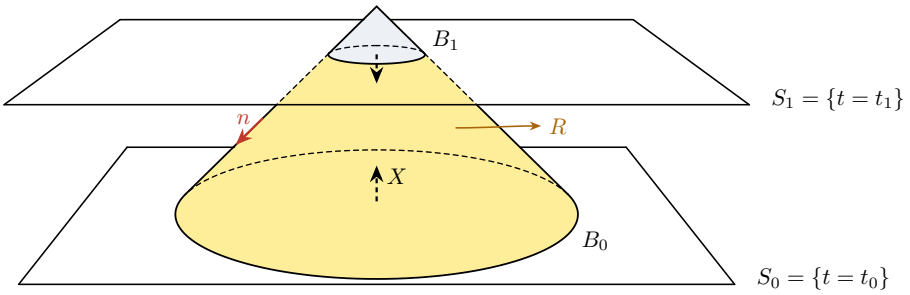


Figure 16: Geometric setting of theorem 2.23.

our solution ϕ to be smooth and compactly supported. The integral over C is non-positive since both J and n are past-pointing (the proof is analogous to that in the proof of theorem 2.20), hence:

$$\int_{B_1} g(J, \partial_t) \leq \int_{B_0} g(J, \partial_t) \leq E[\phi](t_0), \qquad \forall\, t_1 > t_0.$$

By taking the limit $R \to +\infty$:

$$E[\phi](t_1) \leq E[\phi](t_0), \qquad \forall\, t_1 > t_0.$$

The inverse inequality is obtained after noticing that, given a solution $\phi(t, x)$, then $\psi(t, x) = \phi(t_0 + t_1 - t, x)$ is also a solution, and by applying the above procedure to ψ. □

3 Schwarzschild Black Holes

3.1 Definition and Properties

So far, the only metric solution to the Einstein equations that we wrote explicitly was the Minkowski metric m. This is a trivial solution to the Einstein vacuum equations

$$\mathrm{Ric}(g) - \frac{1}{2} R(g) g = 0 \tag{3.1}$$

and, in three spatial dimensions, is given by:

$$\begin{aligned} m &= -dt^2 + dx^2 + dy^2 + dz^2 \\ &= -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \end{aligned}$$

where we are considering spherical coordinates $r = \sqrt{x^2 + y^2 + z^2}$, $\theta \in (0, \pi)$, $\varphi \in [0, 2\pi)$, see fig. 17.

A convenient spacetime diagram can be obtained by suppressing the angular directions. In particular, let $X = X(t, r) \in \mathfrak{X}(\mathbb{R}^{1+3})$ be a spherically

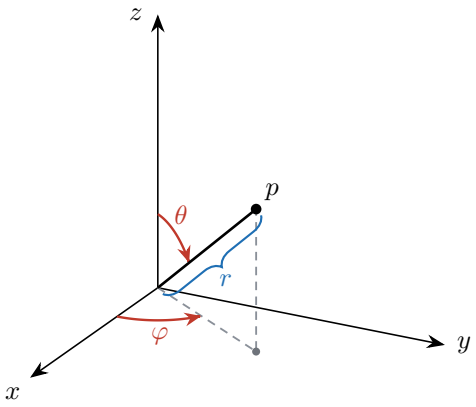


Figure 17: Spherical coordinates.

symmetric vector field. If X is null, this gives a constraint on its components in the current coordinate system:

$$m(X,X) = 0 \text{ if and only if } \big[dt(X)\big]^2 = \big[dr(X)\big]^2.$$

By solving the above, we find that $X_1 = (1,1,0,0)$ and $X_2 = (1,-1,0,0)$ generate the two main null directions (see figure 18).

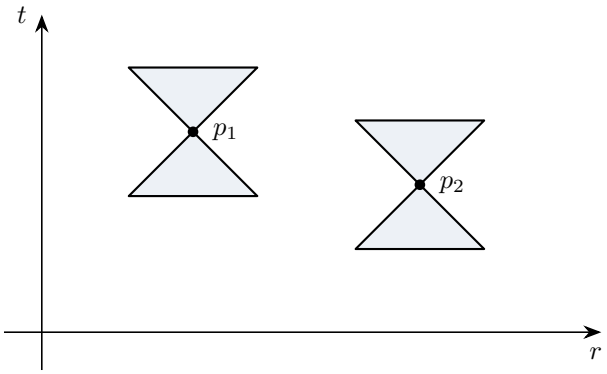


Figure 18: Light cones in Minkowski spacetime.

It turns out that the Einstein equations, even in vacuum (as in (3.1)), admit further solutions which, differently from the Minkowski metric, are non-trivial. Such solutions actually play a paramount role in the theory. Let us consider the following spacetime $(\mathcal{M}, g_M)$, for $M > 0$:

Formally, this reduces to the Minkowski metric when $M = 0$

$$\mathcal{M} = \mathbb{R}_t \times (0,+\infty)_r \times S^2_{\theta,\varphi},$$
$$g_M = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \frac{4M}{r}\,dt\,dr + \Big(1 + \frac{2M}{r}\Big)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

The above $(\mathcal{M}, g_M)$ satisfies the vacuum Einstein equations. It is the *Schwarzschild spacetime*, first presented by Karl Schwarzschild in 1916, one year after the theory of General Relativity was published by Einstein.

To study the main properties of this spacetime, let us investigate its causal structure, i.e. the directions of null vectors.

Let $X = X^t\partial_t + X^r\partial_r = (X^t, X^r)$ be a spherically symmetric vector field in $(\mathcal{M}, g)$, and let us assume that X is null, i.e. $g(X, X) = 0$. This condition implies

$$-\left(1 - \frac{2M}{r}\right)(X^t)^2 + \frac{4M}{r}X^tX^r + \left(1 + \frac{2M}{r}\right)(X^r)^2 = 0. \tag{3.2}$$

For instance:

$$X = (1, -1, 0, 0), \qquad \text{and} \qquad X = \left(1, \frac{1 - \frac{2M}{r}}{1 + \frac{2M}{r}}, 0, 0\right)$$

are linearly independent solutions to (3.2). Note that we have infinitely many solutions to (3.2), since we can always rescale such vectors and still have a solution. The above vector fields give the two main null directions in Schwarzschild spacetime (see fig. 19).

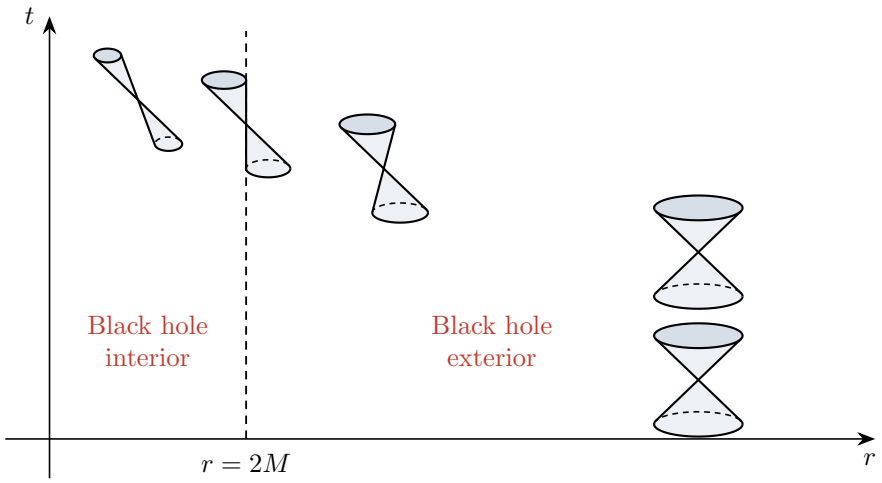


Figure 19: Light cones in Schwarzschild spacetime. Cones are drawn in 3D for illustration purposes. The causal future of any event in $\{0 < r \leq 2M\}$ is still contained in such a region.

Black holes and event horizons

From its causal structure, we see that the causal future of bodies in the $\{r > 2M\}$ region intersects both the $\{r \leq 2M\}$ region and the $\{r > 2M\}$ region. However, observers in $\{r \leq 2M\}$ are forced to stay in such a region, that we call the *black hole interior* region. We say that the Schwarzschild metric describes a *black hole spacetime*. The $\{r = 2M\}$ hypersurface is called *event horizon*.

Schwarzschild solutions, that include the Minkowski case when the mass M is equal to zero, are the most general solutions in spherical symmetry, as expressed by the theorem below.

Remark 3.1 We note that:

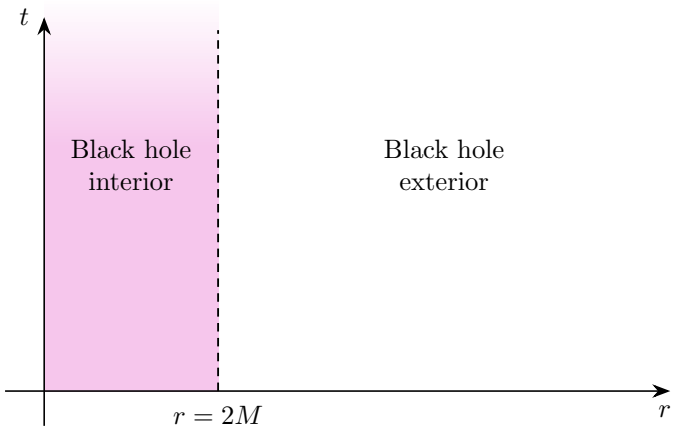


Figure 20: Spacetime diagram of the Schwarzschild spacetime in (t, r) coordinates.

(a) An isometry between two spacetimes $(\mathcal{M}_1, g_1)$, $(\mathcal{M}_2, g_2)$ is a diffeomorphism $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\phi^* g_2 = g_1$ (i.e. such that it preserves the metric tensor), where ϕ^* denotes the pull-back.

Isometry

(b) $(\mathcal{M}, g)$ is a spherically symmetric spacetime if the Lie group $SO(3)$ acts by isometry (i.e. the group action is an isometry) on $\mathcal{M}$ and if the group orbits are either 2-spheres or points.

Spherical symmetry

Theorem 3.2 (Birkhoff, 1923) *Any spherically symmetric solution to the Einstein vacuum equations is locally isometric to a Schwarzschild solution g_M.*

Remark 3.3 Note that there are several coordinate choices to express the Schwarzschild metric. For instance:

Schwarzschild metric in different coordinate systems

$$\begin{aligned} g_M &= -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{4M}{r}\,dt\,dr + \left(1 + \frac{2M}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \\ &= -\left(1 - \frac{2M}{r}\right)d\tilde{t}^2 + \frac{1}{1 - \frac{2M}{r}}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \end{aligned}$$

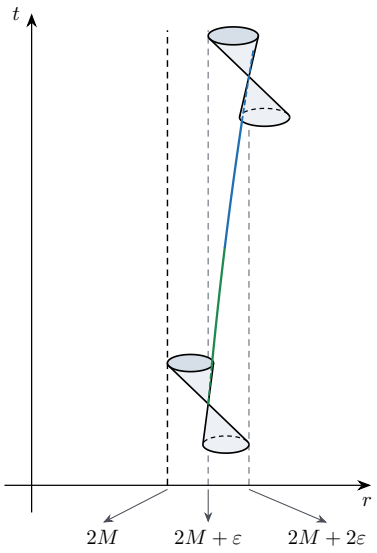
where we went from ingoing Eddington–Finkelstein coordinates to Schwarzschild coordinates (note that the latter are not regular at the event horizon $\mathcal{H}^+ = \{r = 2M\}$), via

$$\tilde{t} = t + f(r), \qquad \text{where } f'(r) = 1 - \frac{1}{1 - \frac{2M}{r}}.$$

3.2 The Redshift Effect

The (gravitational) *redshift effect* (not to be confused with the relativistic Doppler effect) is a phenomenon due to spacetime curvature. Loosely speaking, gravity affects how time is measured in two different locations,

and this implies that, for monochromatic radiation, different wavelengths are measured by static observers at different locations.



Another way to see this is that it is "harder" to run away from a gravitational source, for instance from a black hole. Due to the causal structure in Schwarzschild spacetime, indeed, larger "time" t (compared to Minkowski spacetime) is needed to go from $r_1 = 2M + \varepsilon$ to $r_2 = 2M + 2\varepsilon$ (think of an object travelling from $2M + \varepsilon$ to $2M + 2\varepsilon$).

The above picture, although intuitive, is coordinate-dependent. A more precise construction is the following: take two observers, A and B, moving in Schwarzschild spacetime along

$$\gamma_A(s) = (s, r_A, \theta_0, \phi_0)$$
$$\gamma_B(s) = (s, r_B, \theta_0, \phi_0)$$

respectively, where we are using (t, r, θ, ϕ) coordinates and where $r_A < r_B$ and θ_0, ϕ_0 are fixed (see fig. 21). Suppose that A sends two signals to B and

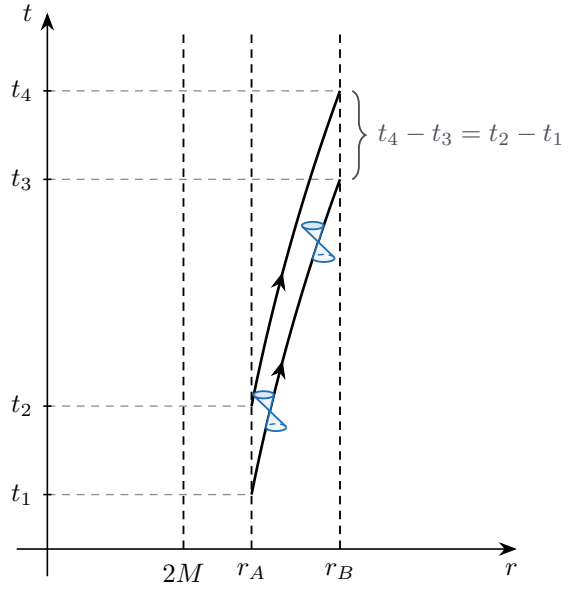


Figure 21: Observers A and B in Schwarzschild spacetime.

that the delay between the two signals, in the t coordinate, is $t_2 - t_1$. The delay between the two signals received by B is $t_2 - t_1$ as well, since the causal structure (the direction of lightcones) is invariant under t-translations. In other terms, the Schwarzschild metric coefficients are independent of t.

The proper time experienced by A between the two signals is:

$$\Delta\tau_A = \int_{t_1}^{t_2} \sqrt{-g(\dot{\gamma}_A(s), \dot{\gamma}_A(s))}\, ds = \sqrt{1 - \frac{2M}{r_A}}\,(t_2 - t_1) \xrightarrow[r_A \to 2M]{} 0.$$

The proper time experienced by B between the two signals is:

$$\Delta\tau_B = \int_{t_3}^{t_4} \sqrt{-g(\dot{\gamma}_B(s), \dot{\gamma}_B(s))}\, ds = \sqrt{1 - \frac{2M}{r_B}}\,(t_2 - t_1).$$

In particular, the time experienced by the observer closer to the black hole is smaller than the one experienced by the observer located farther away:

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \frac{\sqrt{1 - \frac{2M}{r_B}}}{\sqrt{1 - \frac{2M}{r_A}}} \xrightarrow[r_A \to 2M]{} +\infty.$$

It is called REDSHIFT because it denotes an apparent loss of energy when trying to escape from the region close to the black hole event horizon. Indeed, imagine that the two signals are sent to measure the wavelength of a wave (see fig. 22). Then, observer B measures a larger wavelength (associated to smaller energies).

The worldlines of observers A and B are not geodesic, however they are static, meaning that they are integral curves of the Killing vector field ∂_t.

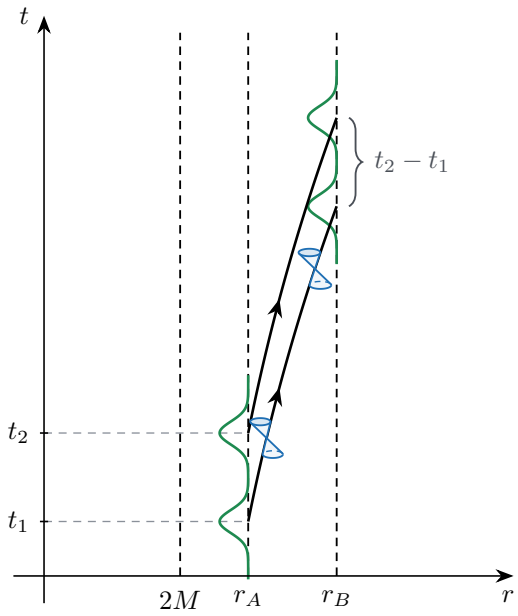


Figure 22: Wavelengths measured by observers A and B.

Remark 3.4 (surface gravity) The surface gravity κ of an event horizon is a constant that is associated to the strength of the redshift effect near such a horizon. Its defining features are summarized by the following results. They hold for time-orientable Lorentzian manifolds that are more general than the Schwarzschild setting we are considering.

Lemma 3.5 *Let $\mathcal{H} \subset \mathcal{M}$ be a null hypersurface. Then, there exists a smooth, non-vanishing null vector field L defined along $\mathcal{H}$ such that $L_p \in T_p\mathcal{H}$ for every $p \in \mathcal{H}$. This L is both tangent and normal to $\mathcal{H}$.*

Lemma 3.6 *There exists a function $f : \mathcal{H} \to \mathbb{R}$ such that*

$$\nabla_L L = fL.$$

Theorem 3.7 (zero-th law of black hole thermodynamics) *If $\mathcal{H}$ is a Killing horizon (i.e. the vector field L of lemma 3.5 can be taken to be a null Killing vector field) in $(\mathcal{M}, g)$, and if g satisfies the Einstein vacuum equations, then the function f of lemma 3.6 can be taken to be a constant, called the surface gravity of $\mathcal{H}$.*

Proof. See the proof of [4, theorem 2.5]. □

This includes the Schwarzschild metric

Remark 3.8 Consider a suitable spacetime $(\mathcal{M}, g)$ with

$$g = -\Omega^2(r)\, dt^2 + \frac{1}{\Omega^2(r)}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

for some function $\Omega(r)$, and let $\mathcal{H} = \{r = r_+\}$ be its event horizon, with $\Omega^2(r_+) = 0$ and $\partial_r\Omega^2(r_+) > 0$. Then the relation

$$\nabla_L L|_{\mathcal{H}} = \kappa L|_{\mathcal{H}} \tag{3.3}$$

for the Killing field $L = \partial_t$ gives:

$$\kappa = \frac{1}{2}(\partial_r\Omega^2)|_{\mathcal{H}} > 0.$$

In particular, the above is obtained after setting $v = t + r^*$, with $dr/dr^* = \Omega^2$, so to obtain:

$$g = -\Omega^2 dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

and after computing the Christoffel symbols in such coordinates. Note that, in the (v, r, θ, φ) coordinate system, we have

$$L = \partial_v.$$

One can prove that null hypersurfaces are ruled by null geodesics. Let λ be the affine parameter of a null geodesic ruling $\mathcal{H}$. Then:

$$\frac{\partial}{\partial\lambda} \text{ is null and geodesic}$$

$$\Rightarrow \frac{\partial}{\partial v} = \kappa(\lambda - \lambda_0)\frac{\partial}{\partial \lambda}, \quad \text{for some } \lambda_0, \text{ by (3.3)}$$
$$\Rightarrow \frac{d\lambda}{dv}\frac{\partial}{\partial \lambda} = \kappa(\lambda - \lambda_0)\frac{\partial}{\partial \lambda}$$
$$\Rightarrow \lambda - \lambda_0 = Ce^{\kappa v}, \quad \text{where } C \text{ depends on the null geodesic}$$
$$\Rightarrow L = \frac{\partial}{\partial v} = e^{\kappa v}\frac{\partial}{\partial \lambda}, \quad \text{after rescaling } \lambda.$$

Now, let U be a unit timelike vector field defined in a neighborhood of $\mathcal{H}$, associated to the trajectories of a family of massive observers. Assume $[L, U] = 0$ (i.e. assume that U is invariant under the flow of L). The energy of the null geodesics, as measured by the family of timelike observers, is:

$$E = -g\left(\frac{\partial}{\partial \lambda}, U\right) = -g\left(e^{-\kappa v}L, U\right).$$

One can prove [4] that:

$$L \cdot E = -\kappa E$$
$$\Rightarrow \quad E = E_0 e^{-\kappa v},$$

that is, redshift is associated to an exponential decay of the energy E.

A family of observers crossing the event horizon of, e.g. a Schwarzschild black hole, measure an exponential decay of the energy

3.3 Further Effects in Schwarzschild Black Hole Exteriors

By the geodesic hypothesis (postulate 1), "massless particles" (e.g. light rays) travel along null geodesics.

Lemma 3.9 *There exist null geodesics in $(\mathcal{M}, g_M)$ living on the timelike hypersurface $\mathcal{PS} = \{r = 3M\}$. Such geodesics are said to be* trapped *at the* photon sphere.

For instance, $\gamma(s) = \left(s, 3M, \frac{\pi}{2}, \frac{s}{3\sqrt{3}M}\right)$ in (t, r, θ, φ) is one of such geodesics (see [2, §5.6.2]).

Such geodesics cannot be radial. A few schematic depictions are given in fig 23.

This is related to an (unstable) concentration of energy on the photon sphere, akin to trapping problems in presence of an obstacle in $\mathbb{R}^{1+n}$

4 The Wave Equation on Schwarzschild Black Holes

Recall the expression of the Schwarzschild metric in ingoing Eddington–Finkelstein coordinates:

$$g_M = -\left(1 - \frac{2M}{r}\right)(dt^*)^2 + \frac{4M}{r}\,dt^*dr + \left(1 + \frac{2M}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

where now we denote the time coordinate by t^* to distinguish it from the Schwarzschild coordinates.

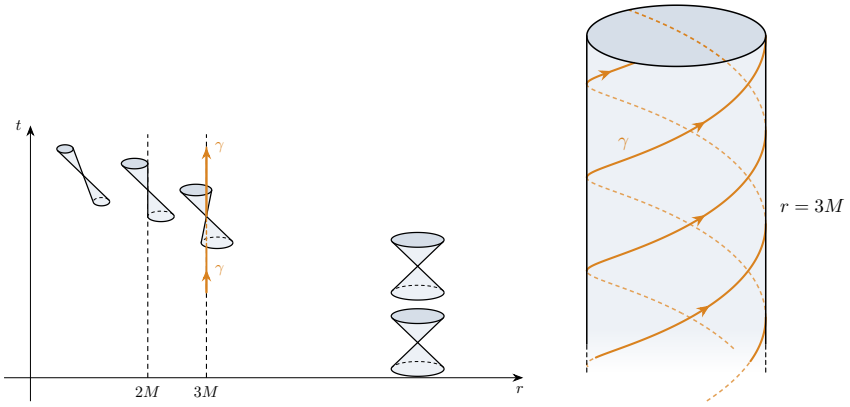


Figure 23: Geodesics on the photon sphere in Schwarzschild spacetime.

Consider the region

$$\mathcal{R} := \{t^* \geq 0,\ r \geq 2M\}.$$

A natural foliation of the Schwarzschild black hole exterior

This is foliated by the spacelike hypersurfaces

$$\Sigma_\tau := \{t^* = \tau,\ r \geq 2M\} \qquad \text{for } \tau \geq 0.$$

Let:

$$T := \partial_{t^*}.$$

Note that the metric coefficients above do not depend on t^* (we say that T is, in particular, a "Killing vector field").

Moreover, let

$$S_\tau := \{t^* = \tau\} \cap \{r = 2M\},$$
$$\mathcal{H}^+ := \bigcup_{\tau \geq 0} S_\tau.$$

For instance, for $\tau_2 > \tau_1$ (angular dimensions are suppressed), see fig 24.

Note that the surfaces Σ_τ, $\tau \geq 0$, are "asymptotically flat", meaning that there exists $K_\tau = \{r \leq R\} \cap \Sigma_\tau$ compact, for some $R > 0$ large, such that $\Sigma_\tau \setminus K_\tau$ is covered by global Euclidean coordinates x^i and the metric $g|_{\Sigma_\tau}$ approaches (in a suitable sense) the flat metric

$$(dx^1)^2 + \cdots + (dx^3)^2.$$

Moreover, the future domain of dependence of Σ_τ (for some $\tau > 0$) is $\{t^* \geq \tau,\ r \geq 2M\}$ (see fig. 25).

Proposition 4.1 *Given $k \in \mathbb{N}$, $k > 1$, $\phi_0 \in H^k_{loc}(\Sigma_0)$, $\phi_1 \in H^{k-1}_{loc}(\Sigma_0)$, there exists a unique solution in $\mathcal{R} = \bigcup_{\tau \geq 0} \Sigma_\tau$ of*

$$\begin{cases} \Box_{g_M} \phi = 0, \\ \phi|_{\Sigma_0} = \phi_0, \\ (n\phi)|_{\Sigma_0} = \phi_1, \end{cases} \tag{WE}$$

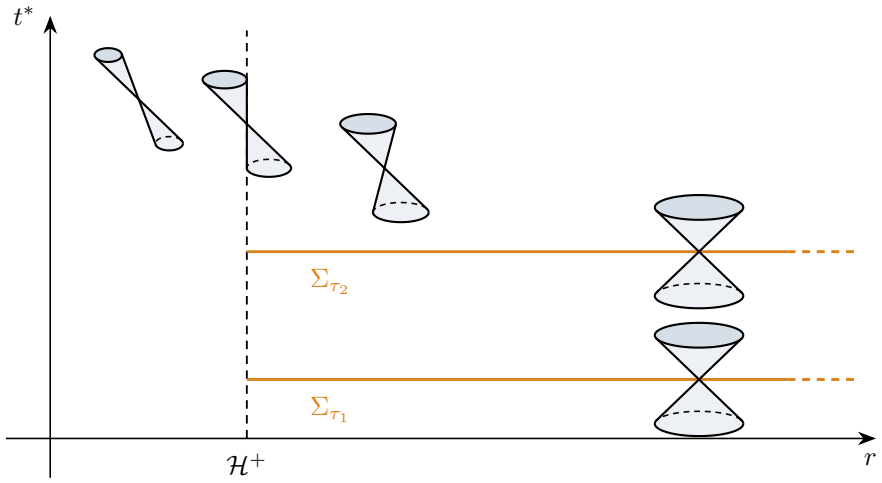


Figure 24: A depiction of Σ_τ hypersurfaces in Schwarzschild spacetime in the (t^*, r) coordinate system. Angular dimensions are suppressed. Cones are drawn in 3D for illustration purposes.

where n is the unit normal to Σ_0, such that for every spacelike hypersurface $S \subset \mathcal{R}$, $\phi|_S \in H^k_{loc}(S)$, $(n\phi)|_S \in H^{k-1}_{loc}(S)$. Moreover, if ϕ_0, ϕ_1 are C^∞, then ϕ is C^∞. If ϕ_0, ϕ_1 have compact support, then $\phi|_{\Sigma_\tau}$, $(n\phi)|_{\Sigma_\tau}$ are compactly supported for every $\tau \geq 0$.

Proof. See [2, Theorem 4.18]. □

Remark 4.2 In practice, the initial data for our solution ϕ determine the solution in the future domain of dependence of the initial hypersurface Σ_0, i.e. in $D^+(\Sigma_0)$. Since $D^+(\Sigma_0)$ is entirely contained in the black hole exterior $\{r \geq 2M\}$, it makes sense to study $\Box_{g_M}\phi = 0$ only in the black hole exterior.[4]

Remark 4.3 In the previous figures we suppressed the angular directions. If we restore them (still suppressing one dimension, since we cannot draw 4 dimensions), i.e. if we rotate around the t^*-axis we obtain a formal schematic depiction as in fig. 26.

Proposition 4.4 (Degenerate energy bound) *Under the previous assumptions, for all $\tau_2 > \tau_1 > 0$, any solution ϕ to* (WE) *satisfies*

$$\int_{\Sigma_{\tau_2}} \left[\left(1 + \frac{2M}{r}\right)(\partial_{t^*}\phi)^2 + \left(1 - \frac{2M}{r}\right)(\partial_r\phi)^2 + |\nabla\!\!\!/\phi|^2\right] r^2\, dr\, d\omega \leq$$
$$\leq \int_{\Sigma_{\tau_1}} \left[\left(1 + \frac{2M}{r}\right)(\partial_{t^*}\phi)^2 + \left(1 - \frac{2M}{r}\right)(\partial_r\phi)^2 + |\nabla\!\!\!/\phi|^2\right] r^2\, dr\, d\omega,$$

where $|\nabla\!\!\!/\phi|^2 = \frac{1}{r^2}\left[(\partial_\theta\phi)^2 + \frac{1}{\sin^2\theta}(\partial_\varphi\phi)^2\right]$ *and* $d\omega = \sin\theta\, d\theta\, d\varphi$.

[4] Note that studying $\Box_{g_M}\phi = 0$ in the black hole interior is an interesting problem as well! The point here is that the interior and the exterior problems can be studied independently due to domain of dependence properties.

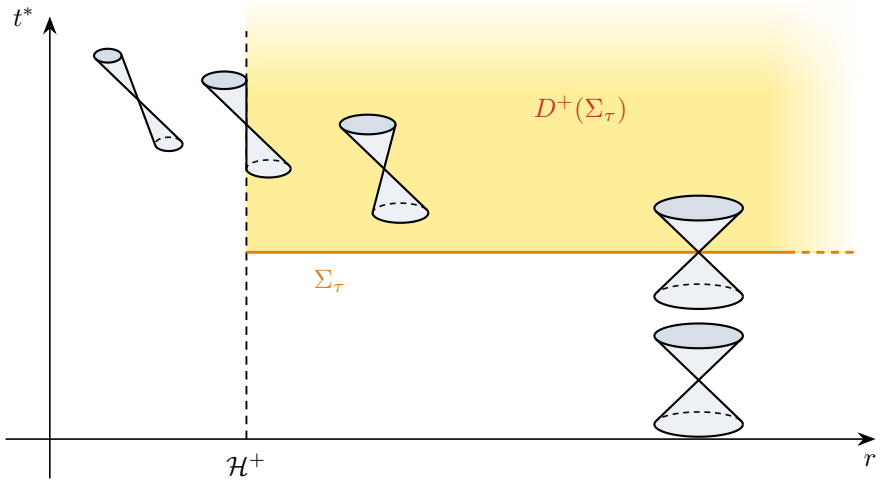


Figure 25: The future domain of dependence of a Σ_τ hypersurface in Schwarzschild spacetime is contained in the black hole exterior region.

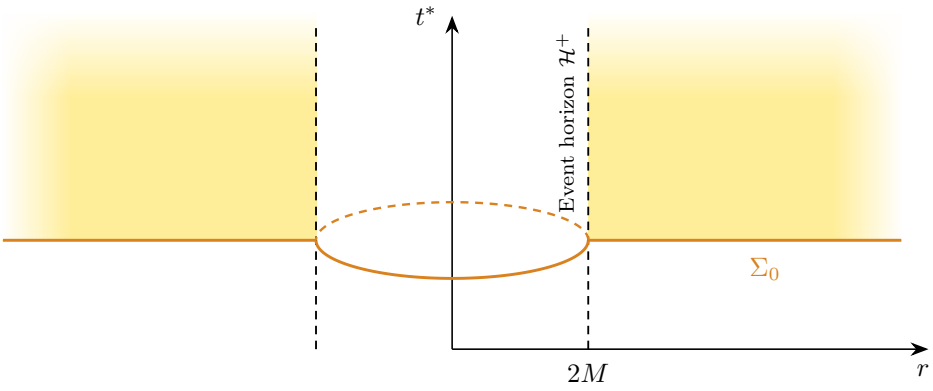


Figure 26: We formally restore one out of two angular directions. From this picture it is more evident that the event horizon of the Schwarzschild spacetime has topology $\mathbb{R} \times S^2$.

Proof. Apply the divergence theorem to the region $\mathcal{R}$ depicted in fig. 27, where $\mathcal{R}$ is the region enclosed by

$$\mathcal{H}^+_{\tau_1,\tau_2} \cup \big(C = \{r = R\}\big) \cup \big(\Sigma_{\tau_2} \cap \{r \leq R\}\big) \cup \big(\Sigma_{\tau_1} \cap \{r \leq R\}\big).$$

Without loss of generality, take the initial data (hence, the solution itself) to be smooth and compactly supported. By compact support, we can choose $R > 0$ large and $C = \{r = R\}$ so that $\phi|_C \equiv 0$. So, with respect to the multiplier vector field $T = \partial_{t^*}$, we have:

$$\int_{\Sigma_{\tau_2}} J^T_\mu[\phi]\, n^\mu = \int_{\Sigma_{\tau_1}} J^T_\mu[\phi]\, n^\mu + \int_{\mathcal{H}^+_{\tau_1,\tau_2}} J^T_\mu[\phi]\, n^\mu_{\mathcal{H}} - \int_{\mathcal{R}} K^T[\phi].$$

Note that $\operatorname{div} J^T[\phi] = K^T[\phi] = 0$, since the metric coefficients are t^*-independent, i.e. $\mathcal{L}_T g = 0$, where $\mathcal{L}$ is the Lie derivative.

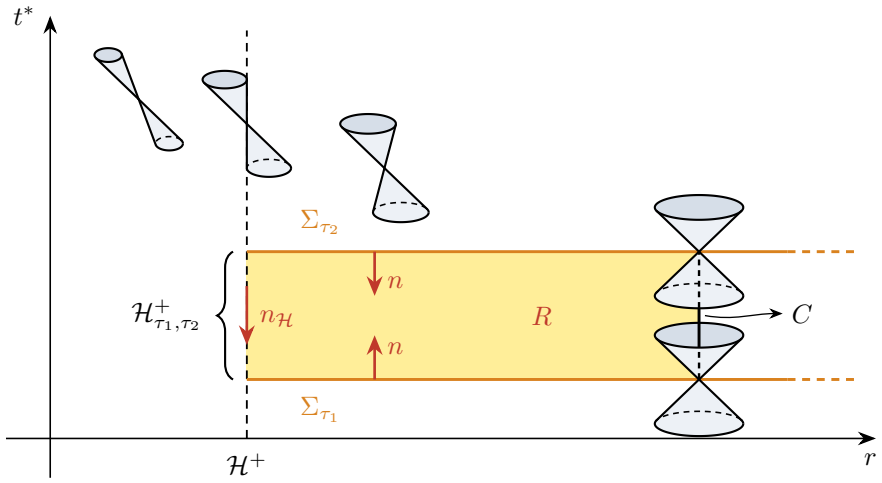


Figure 27: Setting for the proof of proposition 4.4.

Moreover,

$$\int_{\mathcal{H}^+_{\tau_1,\tau_2}} J^T_\mu[\phi]\, n^\mu_{\mathcal{H}} \leq 0,$$

since $g(J,T) \geq 0$ (J is past-pointing) and $g(n_{\mathcal{H}},T) \geq 0$ (the null normal to $\mathcal{H}^+_{\tau_1,\tau_2}$ is past-pointing). So, we are left with:

$$\int_{\Sigma_{\tau_2}} J^T_\mu[\phi]\, n^\mu \leq \int_{\Sigma_{\tau_1}} J^T_\mu[\phi]\, n^\mu,$$

and we just need to write the above expression more explicitly.
Step 1: We first compute the unit timelike vector n, normal to $\Sigma_{\tau_1} = \{t^* = \tau_1\}$. Given the metric g, we write its associated matrix

$$\left(\begin{array}{cc|cc} -\left(1-\frac{2M}{r}\right) & \frac{2M}{r} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \frac{2M}{r} & 1+\frac{2M}{r} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & r^2 & 0 \\ & & 0 & r^2\sin^2\theta \end{array}\right).$$

The inverse matrix is

$$\left(\begin{array}{cc|cc} -\left(1+\frac{2M}{r}\right) & \frac{2M}{r} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \frac{2M}{r} & 1-\frac{2M}{r} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \frac{1}{r^2} & \\ & & & \frac{1}{r^2\sin^2\theta} \end{array}\right).$$

Hence:

$$g^{-1} = -\left(1+\frac{2M}{r}\right)\partial_{t^*}\otimes\partial_{t^*} + \frac{4M}{r}\partial_{t^*}\otimes\partial_r + \left(1-\frac{2M}{r}\right)\partial_r\otimes\partial_r + \\ + \frac{1}{r^2}\left(\partial_\theta\otimes\partial_\theta + \frac{1}{\sin^2\theta}\partial_\varphi\otimes\partial_\varphi\right).$$

Or, in other terms:

$$(g^{-1})^{t^*t^*} = g^{t^*t^*} = -\left(1 + \frac{2M}{r}\right),$$
$$(g^{-1})^{t^*r} = g^{t^*r} = \frac{2M}{r},$$
$$(g^{-1})^{rr} = g^{rr} = 1 - \frac{2M}{r}.$$

Note that the 1-form dt^* is normal to the hypersurface Σ_{τ_1}, meaning that

$$dt^*(Z) = 0 \qquad \forall\, Z \in \mathfrak{X}(\Sigma_{\tau_1}).$$

In layman terms, dt^* selects the t^*-coordinate of Z, but Z has no t^* coordinate since it lives in Σ_{τ_1}. Also note that:

$$g^{-1}(dt^*, dt^*) = -\left(1 + \frac{2M}{r}\right),$$

so

$$\omega := -\frac{dt^*}{\sqrt{1 + \frac{2M}{r}}}$$

is the unit *covector* normal to Σ_{τ_1}. To obtain the corresponding unit normal vector, we use the metric g to raise and lower indices. Given that the components of the covector ω are

$$\omega_\mu = -\frac{1}{\sqrt{1 + \frac{2M}{r}}}\,\delta_\mu^{\ t^*}, \qquad \mu \in \{t^*, r, \theta, \varphi\},$$

then the components of the associated vector $\omega^\sharp$ are

$$\begin{aligned}(\omega^\sharp)^\nu = \omega^\nu = g^{\mu\nu}\omega_\mu &= -\frac{1}{\sqrt{1 + \frac{2M}{r}}}\, g^{\mu\nu}\delta_\mu^{\ t^*} \\ &= -\frac{1}{\sqrt{1 + \frac{2M}{r}}}\, g^{t^*\nu}, \qquad \text{for } \nu \in \{t^*, r, \theta, \varphi\}.\end{aligned}$$

So, given the above expression of the inverse metric g^{-1}:

$$\omega^{t^*} = \sqrt{1 + \frac{2M}{r}},$$
$$\omega^r = -\frac{2M}{r\sqrt{1 + \frac{2M}{r}}}.$$

Therefore:

$$n = \omega = \sqrt{1 + \frac{2M}{r}}\,\partial_{t^*} - \frac{2M}{r\sqrt{1 + \frac{2M}{r}}}\,\partial_r$$

is the unit normal vector to Σ_{τ_1} (one could check that $g(n,n) = -1$).
Step 2: We now compute $J^T_\mu[\phi]\, n^\mu$, with $T = \partial_{t^*}$. For the next computations, it will be useful to calculate

$$\begin{aligned} g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi &= g^{t^*t^*}(\partial_{t^*}\phi)^2 + 2g^{t^*r}\partial_{t^*}\phi\,\partial_r\phi + g^{rr}(\partial_r\phi)^2 + \\ &\quad + g^{\theta\theta}(\partial_\theta\phi)^2 + g^{\varphi\varphi}(\partial_\varphi\phi)^2 \\ &= -\Big(1+\frac{2M}{r}\Big)(\partial_{t^*}\phi)^2 + \frac{4M}{r}\partial_{t^*}\phi\,\partial_r\phi + \Big(1-\frac{2M}{r}\Big)(\partial_r\phi)^2 + \\ &\quad + \frac{1}{r^2}(\partial_\theta\phi)^2 + \frac{1}{r^2\sin^2\theta}(\partial_\varphi\phi)^2. \end{aligned}$$

We define:

$$|\nabla\!\!\!/\phi|^2 := \frac{1}{r^2}\Big[(\partial_\theta\phi)^2 + \frac{1}{\sin^2\theta}(\partial_\varphi\phi)^2\Big],$$

so that:

$$\begin{aligned} g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi &= -\Big(1+\frac{2M}{r}\Big)(\partial_{t^*}\phi)^2 + \frac{4M}{r}\partial_{t^*}\phi\,\partial_r\phi \\ &\quad + \Big(1-\frac{2M}{r}\Big)(\partial_r\phi)^2 + |\nabla\!\!\!/\phi|^2. \end{aligned} \tag{4.1}$$

We now compute:

$$\begin{aligned} J^T_\mu[\phi]\, n^\mu &= T_{\mu\nu}[\phi]\, T^\nu n^\mu \\ &= \Big(\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi\Big)(\partial_{t^*})^\nu\Big(\sqrt{1+\frac{2M}{r}}\,\partial_{t^*} - \frac{2M}{r\sqrt{1+\frac{2M}{r}}}\,\partial_r\Big)^\mu \\ &= \Big(\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi\Big)\delta^\nu{}_{t^*}\Big(\sqrt{1+\frac{2M}{r}}\,\delta^\mu{}_{t^*} - \frac{2M}{r\sqrt{1+\frac{2M}{r}}}\,\delta^\mu{}_r\Big) \\ &\overset{(4.1)}{=} \sqrt{1+\frac{2M}{r}}\,(\partial_{t^*}\phi)^2 - \frac{2M}{r\sqrt{1+\frac{2M}{r}}}\,\partial_{t^*}\phi\,\partial_r\phi + \\ &\quad - \frac{1}{2}\Big(-\Big(1+\frac{2M}{r}\Big)(\partial_{t^*}\phi)^2 + \frac{4M}{r}\partial_{t^*}\phi\,\partial_r\phi + \Big(1-\frac{2M}{r}\Big)(\partial_r\phi)^2 + |\nabla\!\!\!/\phi|^2\Big)\cdot \\ &\quad \cdot \underbrace{\Big(-\Big(1-\frac{2M}{r}\Big)\sqrt{1+\frac{2M}{r}} - \frac{4M^2}{r^2}\cdot\frac{1}{\sqrt{1+\frac{2M}{r}}}\Big)}_{= -\frac{1}{\sqrt{1+\frac{2M}{r}}}} \\ &= (\partial_{t^*}\phi)^2\Big[\sqrt{1+\frac{2M}{r}} - \frac{1}{2}\Big(1+\frac{2M}{r}\Big)\frac{1}{\sqrt{1+\frac{2M}{r}}}\Big] \\ &\quad + \frac{1}{2}(\partial_r\phi)^2\Big(1-\frac{2M}{r}\Big)\frac{1}{\sqrt{1+\frac{2M}{r}}} \\ &\quad + \frac{1}{2}|\nabla\!\!\!/\phi|^2\frac{1}{\sqrt{1+\frac{2M}{r}}} \end{aligned}$$

$$= \frac{1}{2\sqrt{1+\frac{2M}{r}}}\left[\left(1+\frac{2M}{r}\right)(\partial_{t^*}\phi)^2 + \left(1-\frac{2M}{r}\right)(\partial_r\phi)^2 + |\nabla\!\!\!/\phi|^2\right].$$

We note that the term

$$1-\frac{2M}{r} \tag{4.2}$$

vanishes at the event horizon $\mathcal{H} = \{r = 2M\}$. Due to this term, the energy estimate will ultimately be *degenerate* (see already the forthcoming remark 4.5).

Step 3: Now we compute the volume forms. Note that the full form of the integral terms is

$$\int_{\Sigma_{\tau_1}} J^T_\mu[\phi]\, n^\mu\, \sigma,$$

where σ is the induced volume form on Σ_{τ_1}. One way to compute σ is the following. Take an orthonormal frame $\{e_0, e_1, e_2, e_3\}$ in $\mathcal{M}$, for instance

$$\begin{aligned}
e_0 &= n = \sqrt{1+\frac{2M}{r}}\,\partial_{t^*} - \frac{2M}{r\sqrt{1+\frac{2M}{r}}}\,\partial_r,\\
e_1 &= \frac{1}{\sqrt{1+\frac{2M}{r}}}\,\partial_r,\\
e_2 &= \frac{1}{r}\,\partial_\theta,\\
e_3 &= \frac{1}{r\sin\theta}\,\partial_\varphi.
\end{aligned}$$

It is easy to verify that:

$$\begin{aligned}
&g(e_0, e_0) = -1,\\
&g(e_i, e_i) = 1 \quad \forall\, i = 1,2,3,\\
&g(e_\mu, e_\nu) = 0 \quad \text{if } \mu \neq \nu,\ \ \mu,\nu \in \{0,1,2,3\}.
\end{aligned}$$

Then, the following is a volume form on $\mathcal{M}$:

$$e_0^\flat \wedge e_1^\flat \wedge e_2^\flat \wedge e_3^\flat,$$

and $\sigma = \pm\big(e_1^\flat \wedge e_2^\flat \wedge e_3^\flat\big)\big|_{\Sigma_\tau}$ is the induced volume form on Σ_τ, $\tau > 0$ (see also [4, Chapt. 5]), where the sign depends on the orientation.

Now, we have

$$\begin{aligned}
(e_1^\flat)_\mu = g_{\mu\nu}(e_1)^\nu &= \frac{1}{\sqrt{1+\frac{2M}{r}}}\,g_{\mu r}\\
&= \sqrt{1+\frac{2M}{r}}\,\delta_\mu^{\ r} + \frac{2M}{r\sqrt{1+\frac{2M}{r}}}\,\delta_\mu^{\ t^*}.
\end{aligned}$$

So:

$$e_1^\flat = \sqrt{1+\frac{2M}{r}}\,dr + \frac{2M}{r\sqrt{1+\frac{2M}{r}}}\,dt^*$$

and (see previous computations):

$$\begin{aligned} e_0^\flat &= -\frac{dt^*}{\sqrt{1+\frac{2M}{r}}}, \\ e_2^\flat &= r\,d\theta, \\ e_3^\flat &= r\sin\theta\,d\varphi. \end{aligned}$$

We have seen that $dt^* \equiv 0$ on Σ_τ (since $dt^*(Z) = 0\ \forall Z \in \mathfrak{X}(\Sigma_\tau)$), hence:

$$\begin{aligned} \sigma = \left(e_1^\flat \wedge e_2^\flat \wedge e_3^\flat\right)\big|_{\Sigma_\tau} &= \left(\sqrt{1+\frac{2M}{r}}\,dr\right) \wedge (r\,d\theta) \wedge (r\sin\theta\,d\varphi) \\ &= \sqrt{1+\frac{2M}{r}}\,r^2\sin\theta\,dr\wedge d\theta\wedge d\varphi. \end{aligned}$$

Step 4: We now gather all results from the previous steps. We found:

$$\begin{aligned} \int_{\Sigma_{\tau_2}} J_\mu^T[\phi]\,n^\mu\,\sigma = \frac{1}{2}\int_{\Sigma_{\tau_2}} \Big[&\Big(1+\frac{2M}{r}\Big)(\partial_{t^*}\phi)^2 + \\ &+\Big(1-\frac{2M}{r}\Big)(\partial_r\phi)^2 + |\nabla\!\!\!/\phi|^2\Big] r^2\sin\theta\,dr\,d\theta\,d\varphi, \end{aligned}$$

and similarly for $\displaystyle\int_{\Sigma_{\tau_1}} J_\mu^T[\phi]\,n^\mu\,\sigma$.
This yields the desired expression. □

Remark 4.5 (On degeneration) The energy bound of proposition 4.4 is "degenerate" in the sense that it gives no control on $\partial_r\phi$ near the event horizon $\{r = 2M\}$, due to the factor

$$1-\frac{2M}{r}.$$

This factor comes from the coordinate-independent expression

$$g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi$$

inside $T_{\mu\nu}[\phi]\,T^\nu n^\mu$. Since all these terms are fixed, except for the vector field T, one could hope to find a different vector field able to remove such a degeneration. This is not just wishful thinking: we actually expect energy decay near $\mathcal{H}^+ = \{r = 2M\}$ due to the redshift effect analysed in section 3.2. The procedure to remove such a degeneration is sketched in proposition 4.6.

Proposition 4.6 (Non-degenerate energy bound) *Under the previous assumptions, for all $\tau_2 > \tau_1 > 0$, any solution ϕ to (WE) satisfies*

$$\begin{aligned} \int_{\Sigma_{\tau_2}} \Big[(\partial_{t^*}\phi)^2 + (\partial_r\phi)^2 + |\nabla\!\!\!/\phi|^2\Big] r^2\,dr\,d\omega \le \\ \le C\int_{\Sigma_{\tau_1}} \Big[(\partial_{t^*}\phi)^2 + (\partial_r\phi)^2 + |\nabla\!\!\!/\phi|^2\Big] r^2\,dr\,d\omega, \end{aligned}$$

where $|\nabla\!\!\!/\phi|^2 = \frac{1}{r^2}\Big[(\partial_\theta\phi)^2 + \frac{1}{\sin^2\theta}(\partial_\varphi\phi)^2\Big]$, $d\omega = \sin\theta\,d\theta\,d\varphi$ and $C > 0$ is a constant depending only on M and that does not depend on τ_1, τ_2.

Sketch of the proof. The main idea consists of applying the divergence theorem (similarly to the case of the degenerate bound of proposition 4.4), where now we consider a multiplier vector field N having more favourable properties compared to the vector field T.

In particular: there exists a vector field $N \in \mathfrak{X}(\{r \geq 2M\})$ such that

i) N is t^*-independent (i.e. $\mathcal{L}_{\partial_{t^*}} N = 0$, where $\mathcal{L}$ is the Lie derivative),
ii) N is future-directed and timelike,
iii) $N = T$ in $\Sigma_\tau \cap \{r \geq r_1\}$ for some $r_1 > 2M$, for $\tau > 0$,
iv) there exists $r_0 > 2M$ such that, on $\Sigma_\tau \cap \{r \leq r_0\}$, for $\tau > 0$, we have

$$K^N[\phi] \geq c\, J^N_\mu[\phi]\, n^\mu \sim c\Big[(\partial_{t^*}\phi)^2 + (\partial_r \phi)^2 + |\nabla\!\!\!/ \phi|^2\Big]$$

for $c = c(r_0, M) > 0$.

The proof of the above is omitted, but it can be found in [2, prop. 5.4].

Given N, the result then follows by applying the divergence theorem to the N-current of ϕ, and partitioning the spacetime region as

$$\{r \geq 2M\} = \{2M \leq r \leq r_0\} \cup \{r_0 \leq r \leq r_1\} \cup \{r \geq r_1\}.$$

Note that the degenerate bound can be applied to the region $\{r \geq r_1\}$. See [2, Theorem 5.3] for more details on the proof. □

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